Gauss's law in Differential form
and
Poisson's and Laplace's Equations

Gauss's law for electrostatic in differential form ($\nabla . E = \frac{\rho}{\epsilon_0}$)

When the charge is distributed over a volume, if ρ is the density of charge, then the total charge inside the closed surface enclosing the volume will be

$$Q = \iiint_V \rho \, dv$$

The electric flux linked with the surface enclosing this volume, by Gauss's theorem or law is

$$\phi_E = \iint_S \vec{E} . \vec{ds} = \frac{1}{\epsilon_0} \iiint_V \rho \, dv \quad \text{—(1)}$$

But according to Gauss's divergence theorem, the volume integral of divergence of electric field $\vec{E}$ over a volume V is equal to the surface integral of that field $\vec{E}$ over the surface S which encloses the given volume, i.e.,

$$\iiint_V \text{div } \vec{E} \, dv = \iint_S \vec{E} . \vec{ds} \quad \text{—(2)}$$

Thus from Eq^n (1) and (2), we have

$$\iiint_V \text{div } \vec{E} . dv = \frac{1}{\epsilon_0} \iiint_V \rho \, dv$$

Since this is true for any arbitrary volume, the integrands on the two sides must be equal at any point.

Therefore

$$\text{div } E = \frac{\rho}{\epsilon_0}$$

or

$$\boxed{\vec{\nabla} . \vec{E} = \frac{\rho}{\epsilon_0}} \quad \text{—(3)}$$

This is the differential form of Gauss's law and is a fundamental equation of electrostatic. It states that the divergence of electric field $\vec{E}$ at any point is proportional to the charge density at that point.

Poisson's and Laplace's Equations :- The Poisson's and Laplace's equations are the result of combining Gauss's law with the gradient of electric field.

The Gauss's law in differential form is

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{---(1)}$$

where ρ is the volume density of charge.

Electric field $\vec{E}$ is related to a scaler potential function V by,

$$\vec{E} = -\vec{\nabla} V \quad \text{---(2)}$$

Eliminating $\vec{E}$ between Eqn (1) & (2), we get

$$\vec{\nabla}\cdot[-\vec{\nabla} V] = \frac{\rho}{\epsilon_0}$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \quad \text{---(3)} \quad \text{since } \vec{\nabla}\cdot\vec{\nabla} = \nabla^2$$

Eqn (3) is known as Poisson's equation.

This equation gives the value of potential V.

If the region is free of charges or in other words, there is no any kind of charge distribution, i.e. the region of field is free from charge i.e. $\rho = 0$, then eqn (3) reduces to

$$\boxed{\nabla^2 V = 0} \quad \text{---(4)}$$

Eqn (4) is known as Laplace's equation. This equation is valid for throughout the charge free region.

every charge free region.

Mathematically, whole of the electrostatics is the study of Laplace's and Poisson's equations together with the suitable boundary condition because by finding V, E can be obtained.