Plane Sections of Conicoids

All plane sections of a conicoid are conics.

To find the nature of the plane section of a central Conicoid.

Let $ax^2 + by^2 + cz^2 = 1$ be a central conicoid and

let $lx + my + nz = p$ be a plane.

Now $$ax^2 + by^2 + c\left(\frac{p - lx + my}{n}\right)^2 = 1$$

as the generators of a cylinder parallel to z-axis.

After solving we get

$$x^2(an^2 + cl^2) + 2clm\,xy + y^2(bn^2 + cm^2)$$
$$- 2cplx - 2cpmy + (cp^2 - n^2) = 0$$

This conic is the projection of the given section on the plane $z = 0$.

This will be parabola, hyperbola or ellipse if

$$bcl^2 + cam^2 + abn^2 \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$$

Length and direction ratios of the axes of central plane section of a central conicoid.

If $ax^2 + by^2 + cz^2 = 1$ is the central conicoid and $lx + my + nz = 0$ is the central plane and $x^2 + y^2 + z^2 = r^2$ is any concentric sphere.

Then the equation

$$\frac{l^2}{ar^2-1} + \frac{m^2}{br^2-1} + \frac{n^2}{cr^2-1} = 0 \quad \text{——} (1)$$

gives the length of semi axes of the section as equation is quadratic in r^2

Also the equation

$$\frac{\lambda(ar^2-1)}{l} = \frac{\mu(br^2-1)}{m} = \frac{\nu(cr^2-1)}{n}$$

gives the direction ratios of the axis of the length $2r$ where r can be found by (1), and λ, ν, μ are the direction ratios of the axis of length $2r$.

Area of the central section is given by

$$A = \frac{\pi}{p\sqrt{abc}}.$$

ex Show that the section of ellipsoid $9x^2 + 6y^2 + 14z^2$ the plane $x+y+z=0$ is an ellipse with semi axis $\frac{1}{2}$ and $\frac{3}{\sqrt{22}}$.

Sol. Using the eq. (1)

$$\frac{1}{\frac{9r^2}{3}-1} + \frac{1}{\frac{6r^2}{3}-1} + \frac{1}{\frac{14r^2}{3}-1} = 0 \quad \text{i.e.} \quad \frac{1}{(3r^2-1)} + \frac{1}{(2r^2-1)} + \frac{3}{14r^2-3} =$$

after solving we get.

$$(5r^2-2)(14r^2-3)+3(6r^4-5r^2+1)=0$$

$$88r^4-58r^2+9=0$$

$$r^4-\frac{58}{88}r^2+\frac{9}{88}=0$$

ie $\quad r^2=\frac{1}{4}, \quad r^2=\frac{9}{22}$

ie $\quad r=\frac{1}{2}, \quad r=\frac{3}{\sqrt{22}}$

Length and direction ratios of the axes of non central plane section of a central conicoid.

Let the equation of conicoid be. $ax^2+by^2+cz^2=1$

and a non central plane (any plane not passing through the centre) be $lx+my+nz=p\neq 0$

The length of the semi axes then defined by the eq.

$$\frac{l^2}{\frac{ar^2}{k^2}-1}+\frac{m^2}{\frac{br^2}{k^2}-1}+\frac{n^2}{\frac{cr^2}{k^2}-1}=0$$

where $\quad 1-\frac{p^2}{p_0^2}=k \quad$ and $\quad p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$

Direction cosines are given by

$$\frac{\lambda\left(\frac{ar^2}{k^2}-1\right)}{l}=\frac{\mu\left(\frac{br^2}{k^2}-1\right)}{m}=\frac{\nu\left(\frac{cr^2}{k^2}-1\right)}{n}$$

Ex Find the centre, lengths and directions of the axes of the section of the ellipsoid $9x^2+6y^2+14z^2=3$ by the plane $x+y+z=1$

Let (α,β,γ) be the centre of the section of conicoid.

$$3x^2+2y^2+\frac{14}{3}z^2=1 \quad \& \quad x+y+z=1.$$

The plane is represented by $(S_1=T)$

$$3x\alpha+2y\beta+\frac{14}{3}z\gamma=3\alpha^2+2\beta^2+\frac{14}{3}\gamma^2$$

$$\therefore \frac{3\alpha}{1}=\frac{2\beta}{1}=\frac{14\gamma}{3}=\frac{3\alpha^2+2\beta^2+\frac{14}{3}\gamma^2}{1}$$

$$3\alpha=t,\quad 2\beta=t,\quad \frac{14}{3}\gamma=t$$

$$\therefore \frac{21}{22}t^2=t \Rightarrow t=\frac{21}{22}$$

$$\therefore (\alpha,\beta,\gamma)=\left(\frac{7}{22},\frac{21}{44},\frac{9}{44}\right)$$

Now length

$$\frac{1}{3\frac{r^2}{k^2}-1}+\frac{1}{2\frac{r^2}{k^2}-1}+\frac{1}{\frac{14}{3}\frac{r^2}{k^2}-1}=0$$

$$p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=\frac{1}{3}+\frac{1}{2}+\frac{3}{14}=\frac{5}{6}+\frac{3}{14}=\frac{22}{21}$$

$p^2=1$

$$\therefore k=1-\frac{p^2}{p_0^2}=1-\frac{21}{22}=\frac{1}{22} \quad \text{ie } \frac{1}{k}=22$$

$$\text{ie } \frac{1}{3(22r)^2-1}+\frac{1}{2(22r)^2-1}+\frac{1}{\frac{14}{3}(22r)^2-1}=0$$

we get a equation

$$88R^4 - 58R^2 + 9 = 0$$

ie $R^2 = \frac{1}{4}$ and $R^2 = \frac{9}{22}$ $\quad \because R = 22r$

or $22r^2 = \frac{1}{4}$ and $22r^2 = \frac{9}{22}$

$r = \frac{1}{2\sqrt{22}}$, $r = \frac{3}{22}$

Similarly direction can be found by the eq.

$$\frac{\lambda\left(\frac{ar^2}{k^2} - 1\right)}{l} = \frac{\mu\left(\frac{br^2}{k^2} - 1\right)}{m} = \frac{\nu\left(\frac{cr^2}{k^2} - 1\right)}{n}$$

or $\lambda\left(3R^2 - 1\right) = \mu\left(2R^2 - 1\right) = \nu\left(\frac{14R^2 - 3}{3}\right)$

where R has two values mentioned as above.

Circlular Section

The sections of the conicoid by planes are Circles.

ie $F \equiv S + \lambda uv = 0$

where $F = 0$ is conicoid S is sphere, $u=0, v=0$ are planes

Circular Section of an ellipsoid.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ be the ellipsoid.

Then $\left(\frac{x^2 + y^2 + z^2}{a^2} - 1\right) + \left(\frac{y^2}{b^2} - \frac{y^2}{a^2}\right) + \left(\frac{z^2}{c^2} - \frac{z^2}{a^2}\right) = 0$

Now two planes $y^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right) = 0$ —— (1)

cut the ellipsoid where they meet the sphere $S=0$

ie $x^2+y^2+z^2 = a^2$.

Since the section of a sphere by any plane is circle therefore the planes (1) cut the ellipsoid in circles.

<u>Remark</u> - Any two circular section of an ellipsoid of opposite systems lie on a sphere.

While the other planes cut the ellipsoid in circles.

1. $$\frac{x^2+y^2+z^2}{b^2}-1+\left(\frac{x^2}{a^2}-\frac{x^2}{b^2}\right)+\left(\frac{z^2}{c^2}-\frac{z^2}{b^2}\right)=0$$

ie $$\left(\frac{1}{a^2}-\frac{1}{b^2}\right)x^2+\left(\frac{1}{c^2}-\frac{1}{b^2}\right)z^2=0$$

2. $$\frac{x^2+y^2+z^2}{c^2}-1+x^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)=0$$

ie $$\left(\frac{1}{a^2}-\frac{1}{c^2}\right)x^2+\left(\frac{1}{b^2}-\frac{1}{c^2}\right)y^2=0.$$

Find the real circular section of $10x^2-2y^2+z^2+2=0$

$$(x^2+y^2+z^2)+2+9x^2-3y^2=0$$

ie $$9x^2-3y^2=0$$

ie the real circular sections are

$$3x-\sqrt{3}y=\lambda, \quad 3x+\sqrt{3}y=\mu$$

To find the condition that the section of ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane $lx + my + nz = p$

may be a circle.

The real circular central sections of the ellipsoid are given by

$$\frac{x}{a}\sqrt{a^2-b^2} \pm \frac{z}{c}\sqrt{b^2-c^2} = 0$$

which should be identical with the parallel circular section

$(a>b>c)$, $lx + my + nz = 0$

On compairing we get the condition

$$\frac{al}{\sqrt{a^2-b^2}} = \frac{m}{0} = \frac{cn}{\pm\sqrt{b^2-c^2}}$$

Umbilics.

Umbilic is a point circle lying on the conicoid.

If a point $P(\alpha, \beta, \gamma)$ on a conicoid such that the plane parallel to the tangent plane at a point P determine the circular section is called an Umbilic.

The umbilics are the extremities of the diameters passing through the centres of the system of circular sections of an ellipsoid.

To find the Umbilics of an ellipsoid.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ be ellipsoid. At P (α,

its tangent plane is $\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1$

It should be parallel to the central circular sections of the ellipsoid is

$$\frac{x}{a}\sqrt{a^2-b^2}\pm\frac{z}{c}\sqrt{b^2-c^2}=0$$

on comparing.

$$\frac{\alpha}{a\sqrt{a^2-b^2}}=\frac{\beta}{0}=\pm\frac{\gamma}{c\sqrt{b^2-c^2}}=k$$

But P lies on ellipsoid ∴

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=1 \quad\text{—————}\ (1)$$

Substituting the values of α,β,γ in (1)

$$k^2\{(a^2-b^2)+(b^2-c^2)\}=1$$

$$\Rightarrow \quad k=\pm\frac{1}{\sqrt{a^2-c^2}}$$

Hence $\alpha=a\sqrt{\frac{a^2-c^2}{a^2-b^2}}$, $\beta=0$, $\gamma=c\sqrt{\frac{a^2-}{b^2-}}$

These are the coordinates

Sections of Paraboloids

Let the eq. of the paraboloid be $ax^2+by^2=2z$ — ①

and plane be $lx+my+nz=p$. — ②

The equation of cylinder passing through the section of ① by ② and having its generators parallel to x-axis is

$$(am^2+bl^2)y^2+2amnyz+an^2z^2-2apmy$$
$$-2(apn+cl^2)z+ap^2=0.$$

The section is an ellipse, parabola and hyperbola

is $\quad am^2n^2-an^2(am^2+bl^2) <, =, > 0.$

or $\quad abn^2l^2 >, =, < 0.$

1. If $n=0$, then section is parabola.
   If $n\neq 0$ the section will be ellipse if $ab>0$
   or hyperbola if $ab<0$.

i.e. we conclude this.

All the sections of a paraboloid (elliptic or hyperbolic) which are parallel to the axis of parabolas, all other sections of an elliptic paraboloid are ellipses and of a hyperbolic paraboloid are hyperbolas.

Circular section of Paraboloids

we have $\quad ax^2 + by^2 = 2cz$

$$a\left(x^2+y^2+z^2-\frac{2cz}{a}\right)+y^2(b-a)-az^2=0$$

or $\quad b\left(x^2+y^2+z^2-\frac{2cz}{b}\right)+x^2(a-b)-bz^2=0$

Hence two pair of planes

$$y^2(b-a)-az^2=0$$

$$x^2(a-b)-bz^2=0$$

determine the circular section at origin.

<u>Remark</u> → Hyperbolic paraboloids have no real circular section.

<u>ex</u> Find the real circular section of paraboloids.

$$x^2+10y^2=2z$$

$$(x^2+y^2+z^2-2z)+9y^2-z^2=0$$

$$10\left(x^2+y^2+z^2-\frac{2z}{10}\right)+(-9x^2)-10z^2=0$$

Hence the only real circular sections are

$$9y^2-z^2=0$$

or $\quad 3y \pm z = \lambda$

i.e $\quad 3y+z=\lambda, \quad 3y-z=\lambda.$

Umbilics of Paraboloids

Let the equation of paraboloid bye

$$ax^2 + by^2 = 2cz \quad , \quad a > b > 0.$$

The two real circular sections are

$$x\sqrt{a-b} + z\sqrt{b} = \lambda$$

$$x\sqrt{a-b} - z\sqrt{b} = \mu$$

Let (α, β, γ) be an umbilic then tangent plane at (α, β, γ) is

$$a\alpha x + b\beta y = c(\gamma + z)$$

It must be $\parallel$ to the given planes

ie $$\frac{a\alpha}{\sqrt{a-b}} = \frac{b\beta}{0} = \frac{-c}{\pm\sqrt{b}} = k \quad \text{—①}$$

$$\therefore \left[\pm \frac{c}{a}\frac{\sqrt{a-b}}{b} \, , \, 0 \, , \, \frac{(a-b)c}{2ab}\right]$$

are two real umbilic of paraboloid, by using the relation $a\alpha^2 + b\beta^2 = 2c\gamma$ as substituting the value of α, β and γ in this. and find the value of k.

Q. Find the real umbilics of the hyperboloid of two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 . \quad \text{—— (1)}$$

Sol: tangent plane at (α, β, γ) is

$$\frac{\alpha x}{a^2} - \frac{\beta y}{b^2} - \frac{\gamma z}{c^2} = 1 \quad \text{—— (2)}$$

The central circular section of the hyperboloids are

$$\frac{x}{a}\sqrt{a^2+b^2} \pm \frac{z}{c}\sqrt{b^2-c^2} = 0 . \quad \text{—— (3)}$$

on comparing (1) & (3) we get

$$\frac{\alpha/a^2}{\sqrt{a^2+b^2}/a} = \frac{-\beta/b^2}{0} = \frac{-\gamma/c^2}{\pm\sqrt{b^2-c^2}/c} = k$$

$$\alpha = a\sqrt{a^2+b^2}\,k , \quad \beta = 0 , \quad \gamma = c\sqrt{b^2-c^2}\,k$$

(α, β, γ) lie on the hyperboloid then

$$\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - \frac{\gamma^2}{c^2} = 1 \longrightarrow (4)$$

Substituting the value of α, β, γ in (4) we get

$$\left[(a^2+b^2) - (b^2-c^2)\right]k^2 = 1 \Rightarrow k = \pm\frac{1}{\sqrt{a^2+c^2}}$$

$\therefore$ Umbilics are $\quad \pm\frac{a\sqrt{a^2+b^2}}{\sqrt{a^2+c^2}} , 0 , \pm\frac{c\sqrt{b^2-c^2}}{\sqrt{a^2+c^2}}$