Matrices Introduction

Dr. Pragya Mishra
Assistant Professor
Department of Mathematics
Pt. Deen Dayal Upadhaya Govt. Girl's P. G. College
Lucknow

Dr. Vimlesh
Assistant Professor
Department of Mathematics & Statistics,
Sri Ram Swaroop Memorial University,
Lucknow –Deva Road, Barabanki

Matrices Introduction

Matrix algebra has at least two advantages:

•Reduces complicated systems of equations to simple expressions.

•Adaptable to systematic method of mathematical treatment and well suited to computers.

Applications of Matrices

Matrices have many applications in diverse fields of science, commerce and social science. Matrices are used in

a) Computer graphics
b) Optics
c) Cryptography
d) Economics
e) Geology
f) Robotics and Animation
g) Wireless communication and Signal Processing

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Properties:

- **A specified number of rows and a specified number of columns**
- **Two numbers (rows x columns) describe the dimensions or size of the matrix or we say order of the matrix.**

EXAMPLES:

3x3 matrix

2x4 matrix

1x2 matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \end{bmatrix}$$

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

e.g. matrix [A] with elements aij of order mn

$$A_{mn} = \begin{bmatrix} a_{11} & a_{12}\ldots & a_{ij} & a_{in} \\ a_{21} & a_{22}\ldots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

i goes from 1 to m

j goes from 1 to n

Types of Matrices

1. Column matrix or Vector:

The number of rows may be any integer but the number of columns is always 1 and order is $1 \times n$

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

2. Row matrix or Vector

Any number of columns but only one row and of order $n \times 1$

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 10 & 3 & 15 & 2 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix}$$

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns i.e. $m \neq n$

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

4. Square matrix

The number of rows is equal to the number of columns (a square matrix **A** has an order of m x m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 \\ 5 & 9 & 0 \\ 8 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements for which a_{ij} $i=j$

5. Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i = j$

$a_{ij} = 0$ for some or all $i \neq j$

6. Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$ *and* $a_{ij} = 1$ for some or all $i = j$

7. Null (zero) matrix - 0

All elements in the matrix are zero $a_{ij} = 0$ For all i,j

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

8. Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

There are two types of Triangular Matices :

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero i.e. $a_{ij} = 0$ for all $i > j$

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero i.e. $a_{ij} = 0$ for all $i < j$

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

9. Scalar matrix

A diagonal matrix whose main diagonal elements are equal to the same scalar
A scalar is defined as a single number or constant i.e. $a_{ij} = 0$ for all $\boldsymbol{i \neq j}$
$a_{ij} = a$ for all $i = j$

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

10. Sub matrix

A matrix which is obtained from a given matrix by deleting any number of rows or columns is called a sub matrix of the given matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

Then sub matrices of A are $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 5 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$

Matrices

Matrix Operations

Matrices - Operations

Equality of Matrices

Two matrices are said to be equal only when all corresponding elements are equal.

Therefore their size or dimensions are equal as well.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{A} = \mathbf{B}$$

Some Properties of Equality:

- If **A** = **B**, then **B** = **A** for all **A** and **B**
- If **A** = **B**, and **B** = **C**, then **A** = **C** for all **A**, **B** and **C**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

If **A** = **B** then $a_{ij} = b_{ij}$

ADDITION AND SUBTRACTION OF MATRICES

The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted.

COMMUTATIVE LAW:

A + **B** = **B** + **A**

ASSOCIATIVE LAW:

A + (**B** + **C**) = (**A** + **B**) + **C** = **A** + **B** + **C**

EXISTENCE OF ADDITIVE IDENTITY:

A + **0** = **0** + **A** = **A**

EXISTENCE OF ADDITIVE INVERSE:

A + (-**A**) = **0** (where –**A** is the matrix composed of $-a_{ij}$ as elements)

SCALAR MULTIPLICATION OF MATRICES

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then

kA = Ak

Ex. If k=4 and $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$

$$4 \times \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} \times 4 = \begin{bmatrix} 12 & -4 \\ 8 & 4 \\ 8 & -12 \\ 16 & 4 \end{bmatrix}$$

Properties:

- k (**A** + **B**) = k**A** + k**B**
- (k + g)**A** = k**A** + g**A**
- k(**AB**) = (k**A**)**B** = **A**(k)**B**
- k(g**A**) = (kg)**A**

MULTIPLICATION OF MATRICES

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication if the number of columns of **A** must equal the number of rows of **B**

Example. **A** x **B** = **C**

(1x3) (3x1) (1x1)

B x **A** = Not possible!

(2x1) (4x2)

A x **B** = Not possible

(6x2) (6x3)

Example **A** x **B** = **C**

(2x3) (3x2) (2x2)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$
$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$
$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row *i* of **A** with column *j* of **B** – row by column multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1\times 4)+(2\times 6)+(3\times 5) & (1\times 8)+(2\times 2)+(3\times 3) \\ (4\times 4)+(2\times 6)+(7\times 5) & (4\times 8)+(2\times 2)+(7\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Remember also:

IA = A

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix} \quad = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Properties of matrix multiplication

Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:

1. **AI = IA = A** - (existence of multiplicative identity)
2. **A(BC) = (AB)C = ABC** - (associative law)
3. **A(B+C) = AB + AC** - (first distributive law)
4. **(A+B)C = AC + BC** - (second distributive law)

Caution!

1. **AB** not generally equal to **BA**, **BA** may not be conformable
2. If **AB** = **0**, neither **A** nor **B** necessarily = **0**
3. If **AB** = **AC**, **B** not necessarily = **C**

Matrices - Operations

AB not generally equal to **BA**, **BA** may not be conformable

$$T = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 20 \end{bmatrix}$$

$$ST = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 6 \\ 10 & 0 \end{bmatrix}$$

ZERO DIVISOR

If A and B are non zero matrices such that **AB** = **0**, neither **A** nor **B** necessarily equal to zero **, then A and B are called Divisor of zero.**

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Idempotent Matrix: A square matrix having property $A^2 = A$**.**

Nilpotent Matrix: a square matrix with property $A^k = 0$ **where k is least positive integer, called nilpotent matrix of index k.**

Involutory matrix: A square matrix where $A^2 = I$

TRANSPOSE OF A MATRIX

To transpose: Interchange rows and columns

The dimensions of **A**^{T} are the reverse of the dimensions of **A**

$$A_{2\times3} = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}_{2\times3}$$

$$A^T = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}_{3\times2}$$

$$a_{ij} = a^T_{ji}$$ For all i and j

Properties of transposed matrices:

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
2. $(\mathbf{AB})^T = \mathbf{B}^T\,\mathbf{A}^T$
3. $(k\mathbf{A})^T = k\mathbf{A}^T$
4. $(\mathbf{A}^T)^T = \mathbf{A}$

Orthogonal matrix:

A square matrix A with the property $\boldsymbol{A^T A = I}$ is orthogonal matrix.

Now $\boldsymbol{|A^T A| = |I|}$

$$\boldsymbol{|A^T||A| = 1}$$
$$\boldsymbol{|A|^2 = 1}$$

$\boldsymbol{|A| = \pm 1}$ **Thus value of determinany of orthogonal matrix is either 1 or -1.**

Properties of Transpose

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

TRANSPOSE OF MATRIX

$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

SYMMETRIC MATRICES

A Square matrix is symmetric if it is equal to its transpose:

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \qquad A^T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Skew symmetric matrix:

A square matrix is skew symmetric if

$$\mathbf{A} = -\,\mathbf{A}^{\mathrm{T}}$$

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{bmatrix} = -A$$

Remark:

When the original matrix is square, transposition does not affect the elements of the main diagonal

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The identity matrix, **I**, a diagonal matrix **D**, and a scalar matrix, **K**, are equal to their transpose since the diagonal is unaffected.

INVERSE OF A MATRIX

Consider a scalar k. The inverse is the reciprocal or division of 1 by the scalar.

Example:

k=7 the inverse of k or $k^{-1} = 1/k = 1/7$

Division of matrices is not defined since there may be **AB** = **AC** while $\mathbf{B} \neq \mathbf{C}$

Instead matrix inversion is used.

The inverse of a square matrix, **A**, if it exists, is the unique matrix $\mathbf{A}^{-1}$ where:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Inverse of m,atrix

Example:

$$A =_2 A^2 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Because:

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^{-1})^{-1} = A$$

$$(A^{T})^{-1} = (A^{-1})^{T}$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

A square matrix that has an inverse is called a nonsingular matrix

A matrix that does not have an inverse is called a singular matrix

Square matrices have inverses except when the determinant is zero

When the determinant of a matrix is zero the matrix is singular

DETERMINANT OF A MATRIX

To compute the inverse of a matrix, the determinant is required

Each square matrix **A** has a unit scalar value called the determinant of **A**, denoted by det **A** or |**A**|

If

$$A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}$$

then

Matrices - Operations

If **A** = [**A**] is a single element (1x1), then the determinant is defined as the value of the element

Then |**A**| =det **A** = a_{11}

If **A** is (n x n), its determinant may be defined in terms of order (n-1) or less.

Matrices - Operations

MINORS

If **A** is an n x n matrix and one row and one column are deleted, the resulting matrix is an (n-1) x (n-1) submatrix of **A**.

The determinant of such a submatrix is called a minor of **A** and is designated by m_{ij} , where i and j correspond to the deleted

row and column, respectively.

m_{ij} is the minor of the element a_{ij} in **A**.

MATRICES - OPERATIONS

eg.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor

Delete first row and column from **A** .

The determinant of the remaining 2 x 2 submatrix is the minor of a_{11}

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Matrices - Operations

Therefore the minor of a_{12} is:

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Matrices - Operations

COFACTORS

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when $i+j$ is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

Matrices - Operations

DETERMINANTS CONTINUED

The determinant of an n x n matrix **A** can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n}$$

The determinant of **A** is therefore the sum of the products of the elements of the first row of **A** and their corresponding cofactors.

(It is possible to define |**A**| in terms of any other row or column but for simplicity, the first row only is used)

Matrices - Operations

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Has cofactors :

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

And:

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of **A** is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Matrices - Operations

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A| = (3)(2) - (1)(1) = 5$$

Matrices - Operations

For a 3 x 3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

Matrices - Operations

The determinant of a matrix A is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Matrices - Operations

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = (1)(2-0) - (0)(0+3) + (1)(0+2) = 4$$

Matrices - Operations

ADJOINT MATRICES

A cofactor matrix **C** of a matrix **A** is the square matrix of the same order as **A** in which each element a_{ij} is replaced by its cofactor c_{ij} .

Example:

If $$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

The cofactor C of A is $$C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

Matrices - Operations

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adjA = C^T$$

It can be shown that:

A(adj **A**) = (adj**A**) **A** = |**A**| **I**

Example:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$adjA = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Matrices - Operations

$$A(adjA) = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

$$(adjA)A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

MATRICES - OPERATIONS

USING THE ADJOINT MATRIX IN MATRIX INVERSION

Since

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

and

$$\mathbf{A}(\text{adj }\mathbf{A}) = (\text{adj}\mathbf{A})\,\mathbf{A} = |\mathbf{A}|\,\mathbf{I}$$

then

$$A^{-1} = \frac{adjA}{|A|}$$

Matrices - Operations

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$

To check

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrices - Operations

Example 2

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of **A** is

|**A**| = (3)(-1-0)-(-1)(-2-0)+(1)(4-1) = -2

The elements of the cofactor matrix are

$$c_{11} = +(-1), \qquad c_{12} = -(-2), \qquad c_{13} = +(3),$$

$$c_{21} = -(-1), \qquad c_{22} = +(-4), \qquad c_{23} = -(7),$$

$$c_{31} = +(-1), \qquad c_{32} = -(-2), \qquad c_{33} = +(5),$$

Matrices - Operations

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so

$$adjA = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

Matrices - Operations

The result can be checked using

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants

Matrix Inversion

Simple 2 x 2 case

Simple 2 x 2 case

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Since it is known that

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Simple 2 x 2 case

Multiplying gives

$$aw + by = 1$$

$$ax + bz = 0$$

$$cw + dy = 0$$

$$cx + dz = 1$$

It can simply be shown that

$$|A| = ad - bc$$

Simple 2 x 2 case

thus

$$y = \frac{1 - aw}{b}$$

$$y = \frac{-cw}{d}$$

$$\frac{1 - aw}{b} = \frac{-cw}{d}$$

$$w = \frac{d}{da - bc} = \frac{d}{|A|}$$

Simple 2 x 2 case

$$z = \frac{-ax}{b}$$

$$z = \frac{1-cx}{d}$$

$$\frac{-ax}{b} = \frac{1-cx}{d}$$

$$x = \frac{b}{-da+bc} = -\frac{b}{|A|}$$

Simple 2 x 2 case

$$w = \frac{1 - by}{a}$$

$$w = \frac{-dy}{c}$$

$$\frac{1 - by}{a} = \frac{-dy}{c}$$

$$y = \frac{c}{-ad + cb} = -\frac{c}{|A|}$$

Simple 2 x 2 case

$$x = \frac{-bz}{a}$$

$$x = \frac{1 - dz}{c}$$

$$\frac{-bz}{a} = \frac{1 - dz}{c}$$

$$z = \frac{a}{ad - bc} = \frac{a}{|A|}$$

Simple 2 x 2 case

So that for a 2 x 2 matrix the inverse can be constructed in a simple fashion as

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} \dfrac{d}{|A|} & \dfrac{b}{|A|} \\ \dfrac{-c}{|A|} & \dfrac{a}{|A|} \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Exchange elements of main diagonal
- Change sign in elements off main diagonal
- Divide resulting matrix by the determinant

Simple 2 x 2 case

Example

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.3 \\ 0.4 & -0.2 \end{bmatrix}$$

Check inverse

$\mathbf{A}^{-1}$ $\mathbf{A}$=$\mathbf{I}$

$$-\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrices transformation

Rank Of Matrix

ELEMENTARY TRANSFORMATION

1. Interchanging: the interchange of ith row (or columns), denoted by $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$.
2. Scaling: the multiplication of the elements of i^{th} row (or columns), by a nonzero scalar k, denoted by

   $R_i \leftrightarrow kR_j$ or $C_i \leftrightarrow kC_j$.
1. Combining: the addition to (or subtraction from)the elements of i^{th} row (or columns) of k, times the elements of j^{th} row (or columns), denoted by

$R_i \leftrightarrow R_i \pm kR_j$ or $C_i \leftrightarrow C_i \pm kC_j$.

Operation applied on row called row transformation

Operation applied on column is called column operation.

Rank of matrix

The rank of matrix A is the order of any highest order non-vanishing determinant of the matrix.

$$\rho(A) = r \quad or \quad r(A) = r$$

Means there is al least one determinant of order r is not equal to zero and every determinant of order r+1 is zero

Ex: A $= \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ $\qquad |A| = \begin{vmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 1 & 2 & 2 \end{vmatrix} = 0$

So rank of A cannot be 3

Now $\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} \neq 0$ (nonzero determinant)

Thus $\quad \rho(A) = 2$

Methods for finding rank of matrix

1. Finding the largest order non-vanishing determinant of matrix A.

2. Reducing Matrix A to Echelon Form.

3. Reducing Matrix A to Normal Form.

Echelon Form:

- All the zero rows or any zero rows follows the nonzero rows.
- The number of zeros before the first non-zero element in first, second, third row should be in increasing order.

The rank of matrix in Echelon form is equal to the number of non zero rows of the matrix.

Ex:

$$A=\begin{bmatrix}1&2&1&2\\1&3&2&2\\2&4&3&4\\3&7&4&6\end{bmatrix}\quad R_2\to R_2-R_1\qquad R_3\to R_3-2R_1$$

$$and\quad R_4\to R_4-3R_1$$

$$A\approx\begin{bmatrix}1&2&1&2\\0&1&1&0\\0&0&1&0\\0&1&1&0\end{bmatrix}\qquad R_4\to R_4-R_2$$

$$\approx\begin{bmatrix}1&2&1&2\\0&1&1&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

Hence $\rho(A)=3$

Normal Form of a matrix:

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Where I_r is an identity matrix of order r and 0 is zero-matrix by e-transformation. Hence rank of matrix is r.

Ex.

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix} \quad R_2 \to R_2 - R_1 \quad R_3 \to R_3 - 3R_1$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} C_2 \to C_2 - C_1 \quad C_3 \to C_3 - C_1 \quad C_4 \to C_4 + C_1$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad R_3 \to R_3 - R_2 \quad C_3 \to C_3 - 2C_2$$

$$C_4 \to C_4 + 5C_2$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\approx I_2$$

$$Hence \quad \rho(A) = 2$$

Inverse of Matrix By E-row operations

Ex. If $A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ find inverse of A.

write A=I A

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

On performing suitable E-row 0perations on the left and on the prefactor of Aon the right till we get I_3 on the left.

$$R_3 \leftrightarrow R_2, R_1 \to R_1 - 2R_2, R_3 \to -R_3, R_2 \to R_2 + 2R_3$$

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & -3 \\ 0 & 1 & -2 \end{bmatrix} A$$

$$R_1 \to R_1 - 4R_2, R_3 \to -R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} A$$

$$\mathrm{I} = \mathrm{BA}$$

$$A^{-1} = B = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}$$

Matrices and Linear Equations

Linear Equations

Linear Equations

Linear equations are common and important for survey problems

Matrices can be used to express these linear equations and aid in the computation of unknown values.

System of equations has three types of solution:

Unique Solutions: consider the system of equations

$$x + 2y = 5$$
$$3x - y = 1$$

$which\ gives\ x = 1, y = 2$ have a single solution or unique solution and said to be consistent by nature.

Infinite solution: Consider

$$x + 2y = 5$$
$$2x + 4y = 10$$

Which does not gives unique solution, but there are many solutions if $x = k$ and $y = 5 - 2k$

k is arbitrary have infinite number of values , so above equation have infinite solutions and said to be consistent.

No Solution: consider

$$x + 2y = 5$$
$$2x + 4y = 7$$

We get $0 = -3$ which absurd therefore equations are said to inconsistent and have no solution.

There are Two types of linear Equations

1. Non Homogenous Linear Equation: AX=B
2. Homogenous Linear Equation: AX=0

Example

- n equations in n unknowns, the a_{ij} are numerical coefficients, the b_i are constants and the x_j are unknowns which is non-homogeneous equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

- The equations may be expressed in the form

$$\mathbf{AX} = \mathbf{B}$$

$$A=\begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n1}\cdots & a_{nn} \end{bmatrix}_{n \times n}, X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, \text{and} \quad B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1}$$

Number of unknowns = number of equations = n

Augmented Matrix: The matrix composed of mn elements of coefficient matrix A plus one addition column whose elements are constant b_i is called augmented matrix of the system and denoted [A, B]

$$[A,B]=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

Solution Of non Homogenous Equation:

If There are n Equations in n Variables

a) If $\rho(A) = \rho[A, B] = r = n$(number of variables), the system will have Unique solution.

b) If $\rho(A) = \rho[A, B] = r < n$, the system will have infinite solution.

c) If $\rho(A) \neq \rho[A, B]$ the system will have no solution.

Example: solve the system of equation

$$2x + y - 2z = 2$$
$$x + y + z = 4$$
$$3x - y + z = 2$$
$$x + 2y + 2z = 7$$

Here

$$\begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \\ 3 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 7 \end{bmatrix} \qquad AX=B$$

Augmented Matrix

$$[A,B]=\begin{bmatrix} 2 & 1 & -2 & \dots & 2 \\ 1 & 1 & 1 & \dots & 4 \\ 3 & -1 & 1 & \dots & 2 \\ 1 & 2 & 2 & \dots & 7 \end{bmatrix} \quad \mathrm{R}_1 \leftrightarrow \mathrm{R}_2$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & \dots & 4 \\ 2 & 1 & -2 & \dots & 2 \\ 3 & -1 & 1 & \dots & 2 \\ 1 & 2 & 2 & \dots & 7 \end{bmatrix} \quad [\mathrm{R}_2 \to \mathrm{R}_2 - 2\mathrm{R}_1, \mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_1$$

$$\mathrm{R}_4 \to \mathrm{R}_4 - \mathrm{R}_1]$$

$$[\mathrm{R}_2 \leftrightarrow \mathrm{R}_4] \quad [\mathrm{R}_3 \to \mathrm{R}_3 + 4\mathrm{R}_2, \mathrm{R}_4 \to \mathrm{R}_4 + \mathrm{R}_2]$$

$$[\mathrm{R}_4 \to -\frac{1}{3}\mathrm{R}_4, \mathrm{R}_3 \to \frac{1}{2}\mathrm{R}_3]$$

$$[\mathrm{R}_4 \to \mathrm{R}_4 - \mathrm{R}_3]$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & \dots & 4 \\ 0 & 1 & 1 & \dots & 3 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Which is Echelon form of Matrix [A,B]

$\rho(A) = \rho[A, B]$ therefore equations are consistent.

Now $\rho(A) = \rho[A, B] = 3 = n$(number of variables), the system will have Unique solution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$x + y + z = 4$$
$$y + z = 3$$
$$z = 1$$

which gives $x = 1, y = 2 \; and \; z = 1.$

Example: Solve the equations such that equations have (i) no Solution (ii) a Unique solution (iii) a Infinite solution

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + az = b$$

Here $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ b \end{bmatrix}$

Then augmented matrix

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & \ldots & 6 \\ 1 & 2 & 3 & \ldots & 10 \\ 1 & 2 & a & \ldots & b \end{bmatrix}$$

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_2$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & \ldots & 6 \\ 0 & 1 & 2 & \ldots & 4 \\ 0 & 0 & a-3 & \ldots & b-10 \end{bmatrix}$$

Case I: when $a = 3, b \neq 10$ we get

$$\rho(A) = 2 \quad \rho[A, B] = 3$$

Thus system is inconsistent and have no solution.

Case II: when $a = 3, b = 10$ we get

$$\rho(A) = \rho[A, B] = 2 < 3(No. of\ variables)$$

Thus system is consistent and infinite solution.

Case III: when $a \neq 3$ we get

$$\rho(A) = \rho[A, B] = 3(No. of\ variables)$$

Thus system is consistent and have unique solution.

SOLUTION IF HOMOGENOUS LINEAR EQUATION

when the system has n Equations in n variables

The equation AX=0 will always have unique solution if rank of A is always equal to number of variables, but it is zero solution also Known as Trivial solution. In this case A is non singular matrix i.e. $|A| \neq 0$.

When rank of matrix A is r less than number of variables then there will be infinite solutions

In this case $|A| = 0$ and we have non zero solution called non trivial solution.

Example: Solve

$$x - y + z = 0$$
$$x + 2y - z = 0$$
$$2x + y - 3z = 0$$

matrix equation becomes $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ AX=0

The Coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & -3 \end{bmatrix}$$

$$[R_2 \to R_2 - R_1, R_3 \to R_3 - 2\,R_1] \text{ and then}$$

$$[R_3 \to R_3 - R_2]$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

This is Echelon form and rank of A =3 (No. of variables). Therfore zero solution is the only solution.

Now matrix equation

$$x - y + z = 0$$

$$3y - 2z = 0$$

$$3z = 0$$

Which gives x=0,y=0 and z=0.

Solution of linear equation by matrix inversion method

If the determinant is nonzero, the equation can be solved to produce n numerical values for x that satisfy all the simultaneous equations

To solve, premultiply both sides of the equation by $\mathbf{A}^{-1}$ which exists because $|\mathbf{A}| \neq \mathbf{0}$

$$\mathbf{A}^{-1}\,\mathbf{AX} = \mathbf{A}^{-1}\,\mathbf{B}$$

Now since

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

We get

$$\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$$

So if the inverse of the coefficient matrix is found, the unknowns, **X** would be determined

Linear Equations

Example

$$3x_1 - x_2 + x_3 = 2$$
$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 - x_3 = 3$$

The equations can be expressed as

$$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Linear Equations

When $\mathbf{A}^{-1}$ is computed the equation becomes

$$X = A^{-1}B = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$$

Therefore

$$x_1 = 2,$$
$$x_2 = -3,$$
$$x_3 = -7$$

Linear Equations

The values for the unknowns should be checked by substitution back into the initial equations

$$x_1 = 2, \quad x_2 = -3, \quad x_3 = -7$$

$$3x_1 - x_2 + x_3 = 2$$
$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 - x_3 = 3$$

$$3\times(2)-(-3)+(-7)=2$$
$$2\times(2)+(-3)=1$$
$$(2)+2\times(-3)-(-7)=3$$