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AS-203

ENGINEERING MATHEMATICS-II

Fourier Series

EULER'S FORMULAE:

The Fourier series for the function f(x) in the interval $\alpha < x < \alpha + 2\pi$ *is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \end{aligned}\right\}$$

These values of a_0, a_n, b_n are known as *Euler's formulae***.

Cor. 1. Making $\alpha = 0$, *the interval becomes* $0 < x < 2\pi$, and the formulae (I) reduce to

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right\}$$

Cor. 2. Putting $\alpha = -\pi$, *the interval becomes* $-\pi < x < \pi$ and the formulae (I) take the form :

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}\right\}$$

DIRICHLET'S CONDITION:

Any function f(x) can be developed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ *where* a_0, a_n, b_n *are constants, provided :*

(*i*) *f* (*x*) *is periodic, single-valued and finite;*

(*ii*) *f* (*x*) *has a finite number of discontinuties in any one period;*

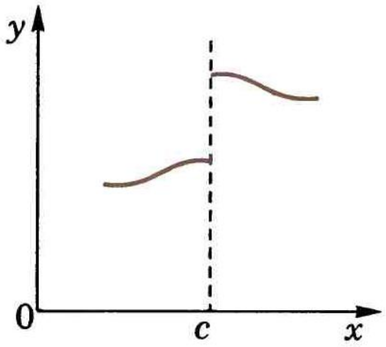
(*iii*) *f* (*x*) *has at the most a finite number of maxima and minima.* (*Anna, 2009 ; P.T.U., 2009*)

In fact the problem of expressing any function *f* (*x*) as a Fourier series depends upon the evaluation of the integrals.

$$\frac{1}{\pi}\int f(x)\cos nx\,dx \,;\, \frac{1}{\pi}\int f(x)\sin nx\,dx$$

within the limits $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$ according as *f* (*x*) is defined for every value of *x* in $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$.

- **At the point of discontinuity**



i.e., at $x = c$, $\quad f(x) = \frac{1}{2}[f(c-0) + f(c+0)]$.

QUESTION:

Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

Solution. Let $$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ..$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left| -e^{-x}\right|_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$$

and $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\left| e^{-x}(-\cos nx + n\sin nx)\right|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{n^2+1}$$

$\therefore$ $$a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\; a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{5} \text{ etc.}$$

Finally, $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\left| e^{-x}(-\sin nx - n\cos nx)\right|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{n}{n^2+1}$$

$\therefore$ $$b_1 = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{1}{2},\; b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{2}{5} \text{ etc.}$$

Substituting the values of a_0, a_n, b_n in (i), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\right)\right\}.$$

Example 10.5. *Find the Fourier series expansion for $f(x)$, if*

$$f(x) = -\pi, \quad -\pi < x < 0$$
$$x, \quad 0 < x < \pi.$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Solution. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(*i*)

Then
$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\,dx + \int_0^{\pi} x\,dx\right] = \frac{1}{\pi}\left[-\pi \,|\, x \,|_{-\pi}^{0} + \left| x^2/2 \right|_0^{\pi}\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = -\frac{\pi}{2};$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$
$$= \frac{1}{\pi}\left[-\pi\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right|_0^{\pi}\right]$$
$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1)$$

$\therefore$
$$a_1 = \frac{-2}{\pi \,.\, 1^2},\ a_2 = 0,\ a_3 = -\frac{2}{\pi\,.\,3^2},\ a_4 = 0,\ a_5 = -\frac{2}{\pi\,.\,5^2} \text{ etc.}$$

Finally,
$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$
$$= \frac{1}{\pi}\left[\left|\frac{\pi\cos nx}{n}\right|_{-\pi}^{0} + \left|-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right|_0^{\pi}\right]$$
$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$\therefore$
$$b_1 = 3,\ b_2 = -\frac{1}{2},\ b_3 = 1,\ b_4 = -\frac{1}{4}, \text{ etc.}$$

Hence substituting the values of a's and b's in (*i*), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots \quad \ldots(ii)$$

which is the required result.

Putting $x = 0$ in (*ii*), we obtain $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \infty\right)$ ...(*iii*)

Now $f(x)$ is discontinuous at $x = 0$. As a matter of fact

$$f(0-0) = -\pi \text{ and } f(0+0) = 0 \quad \therefore \quad f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\pi/2.$$

Hence (*iii*) takes the form $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$ whence follows the result.

CHANGE OF INTERVAL:

Fourier expansion of $f(x)$ in the interval $(\alpha, \alpha + 2c)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx \\ a_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx \\ b_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx \end{aligned}\right\} \quad \text{...(4)}$$

Cor. Putting $\alpha = 0$ in (4), we get the results for the interval $(0, 2c)$ and putting $\alpha = -c$ in (4), we get results for the interval $(-c, c)$.

QUESTION:

Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.

Solution. The required series is of the form

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ...(i)$$

Then $$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx = \frac{1}{l}\left|-e^{-x}\right|_{-l}^{l} = \frac{1}{l}(e^{l} - e^{-l}) = \frac{2\sinh l}{l}$$

and $$a_n = \frac{1}{l}\int_{-l}^{l} e^{x} \cos\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2l(-1)^n \sinh l}{l^2+(n\pi)^2} \qquad [\because \cos n\pi = (-1)^n$$

$$\therefore \quad a_1 = \frac{-2l\sinh l}{l^2+\pi^2},\ a_2 = \frac{2l\sinh l}{l^2+2^2\pi^2},\ a_3 = \frac{2l\sinh l}{l^2+3^2\pi^2} \text{ etc.}$$

Finally, $$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2n\pi(-1)^n \sinh l}{l^2+(n\pi)^2}$$

$$\therefore \quad b_1 = \frac{-2\pi\sinh l}{l^2+\pi^2},\ b_2 = \frac{4\pi\sinh l}{l^2+2^2\pi^2},\ b_2 = \frac{-6\pi\sinh l}{l^2+3^2\pi^2} \text{ etc.}$$

Substituting the values of a's and b's in (i), we get

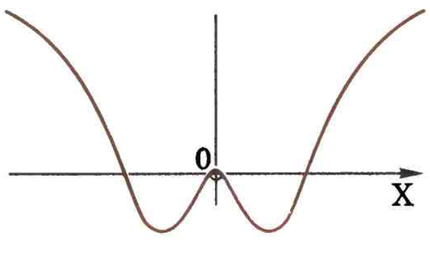
$$e^{-x} = \sinh l\left\{\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - \dots\right)\right.$$
$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{n\pi}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \dots\right)\right\}$$

EVEN AND ODD FUNCTION:

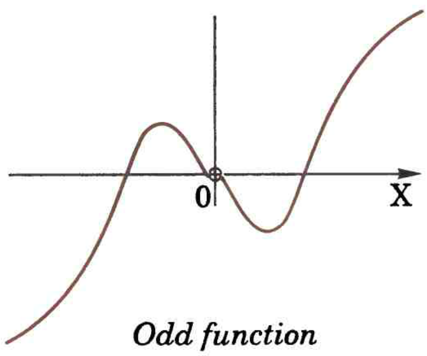
A function $f(x)$ is said to be **even** *if $f(-x) = f(x)$,*

e.g., $\cos x$, $\sec x$, x^2 are all even functions. Graphically an even function is symmetrical about the y-axis.

A function $f(x)$ is said to be **odd** *if $f(-x) = -f(x)$,*



Even function



Odd function

e.g. sin x, tan x, x^3 are odd functions. Graphically, an odd function is symmetrical about the origin.

We shall be using the following property of definite integrals in the next paragraph :

$$\int_{c}^{c} \mathbf{f(x)dx} = 2\int_{0}^{c} \mathbf{f(x)dx}, \quad \textit{when } f(x) \textit{ is an even function.}$$

$$= \mathbf{0}, \ \textit{when } f(x) \textit{ is an odd function.}$$

(2) Expansions of even or odd periodic functions. We know that a periodic function $f(x)$ defined in $(-c, c)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c},$$

where $$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx, \ a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx, \ b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx.$$

Case I. *When f(x) is an even function* $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{2}{c}\int_{0}^{c} f(x)dx.$

Since $f(x)\cos\frac{n\pi x}{c}$ is also an even function,

$$\therefore \quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = \frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}dx$$

Again since $f(x)\sin\frac{n\pi x}{c}$ is an odd function, $\therefore \ b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = 0.$

Hence, *if a periodic function f (x) is even, its Fourier expansion contains only cosine terms, and*

$$\left.\begin{aligned} a_0 &= \frac{2}{c}\int_{0}^{c} f(x)dx \\ a_n &= \frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}dx \end{aligned}\right\}$$

Case II. *When f(x) is an odd function,* $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = 0,$

Since $\cos\frac{n\pi x}{c}$ is an even function, therefore, $f(x)\cos\frac{n\pi x}{c}$ is an odd function.

$$\therefore \quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c} = 0$$

Again since $\sin\frac{n\pi x}{c}$ is an odd function, therefore, $f(x)\sin\frac{n\pi x}{c}$ is an even function.

$$\therefore \quad b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{2}{c}\int_{0}^{c} f(x)\sin\frac{n\pi x}{c}dx$$

Thus, *if a periodic function f (x) is odd, its Fourier expansion contains only sine terms and*

$$b_n = \frac{2}{c}\int_{0}^{c} f(x)\sin\frac{n\pi x}{c}dx$$

QUESTION:

Find a Fourier series to represent x^2 in the interval $(-l, l)$.

Solution. Since $f(x) = x^2$ is an even function in $(-l, l)$,

$$\therefore \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Then
$$a_0 = \frac{2}{l}\int_0^l x^2\,dx = \frac{2}{l}\left|\frac{x^3}{3}\right|_0^l = \frac{2l^2}{3}$$

$$a_n = \int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x^2\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 2x\left(-\frac{\cos n\pi x/l}{n^2\pi^2/l^2}\right) + 2\left(-\frac{\sin n\pi x/l}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= 4l^2(-1)^n/n^2\pi^2$$

$$\therefore \qquad a_1 = -4l^2/\pi^2,\ a_2 = 4l^2/2^2\pi^2,\ a_3 = -4l^2/3^2\pi^2,\ a_4 = 4l^2/4^2\pi^2 \text{ etc.}$$

Substituting these values in (i), we get

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\frac{\cos \pi x/l}{1^2} - \frac{\cos 2\pi x/l}{2^2} + \frac{\cos 3\pi x/l}{3^2} - \frac{\cos 4\pi x/l}{4^2} + \ldots\right)$$

HALF RANGE SINE AND COSINE SERIES:

Sine series. If it be required to expand $f(x)$ as a sine series in $0 < x < c$; then we extend the function reflecting it in the origin, so that $f(x) = -f(-x)$.

Then the extended function is odd in $(-c, c)$ and the expansion will give the desired Fourier sine series :

$$\left.\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \\ \text{where} \quad b_n &= \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c}\, dx \end{aligned}\right\} \quad \text{...(1)}$$

Cosine series. If it be required to express $f(x)$ as a cosine series in $0 < x < c$, we extend the function reflecting it in the y-axis, so that $f(-x) = f(x)$.

Then the extended function is even in $(-c, c)$ and its expansion will give the required Fourier cosine series :

$$\left.\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} \\ \text{where} \quad a_0 &= \frac{2}{c} \int_0^c f(x)\, dx \\ \text{and} \quad a_n &= \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi x}{c}\, dx \end{aligned}\right\} \quad \text{...(2)}$$

QUESTION:

Obtain the Fourier expansion of x sin x as a cosine series in (0, π).

Hence show that $\dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \ldots \infty = \dfrac{\pi - 2}{4}$.

Solution. Let $x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \frac{2}{\pi} \int_0^{\pi} x \sin x \, dx = \frac{2}{\pi} \left| x(-\cos x) - 1(-\sin x) \right|_0^{\pi} = 2$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx = \frac{1}{\pi} \int_0^{\pi} x (\sin \overline{n+1}\, x - \sin \overline{n-1}\, x) \, dx$$

$$= \frac{1}{\pi} \left[x \left\{ \frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right\} - 1. \left\{ \frac{-\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \pi \left\{ \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} \right\} (n \neq 1).$$

When $n = 1$, $a_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx$

$$= \frac{1}{\pi} \left| x \left(\frac{-\cos 2x}{2} \right) - 1 \left(\frac{-\sin 2x}{2} \right) \right|_0^{\pi} = \frac{1}{\pi} \left(-\frac{\pi \cos 2\pi}{2} \right) = -\frac{1}{2}.$$

Hence $x \sin x = 1 - \frac{1}{2} \cos x - 2 \left\{ \frac{\cos 2x}{1.3} - \frac{\cos 3x}{3.5} + \frac{\cos 4x}{5.7} - \dots \infty \right\}$

Putting $x = \pi/2$, we obtain $\pi/2 = 1 + 2 \left\{ \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots \infty \right\}$

Hence $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots \infty = \frac{\pi - 2}{4}$.

QUESTION:

Express f(x) = x as a half-range sine series in 0 < x < 2.

SOLUTION: By the formula of half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

where $b_n = \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} \, dx = \int_0^2 x \sin \frac{n\pi x}{2} \, dx$

$$= \left| -\frac{2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right|_0^2 = -\frac{4(-1)^n}{n\pi}$$

Thus $b_1 = 4/\pi$, $b_2 = -4/2\pi$, $b_3 = 4/3\pi$, $b_4 = -4/4\pi$ etc.

Hence the Fourier sine series for $f(x)$ over the half-range (0, 2) is

$$f(x) = \frac{4}{\pi} \left(\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} - \frac{1}{4} \sin \frac{4\pi x}{2} + \dots \right).$$

SOURCE: HIGHER ENGINEERING MATHEMATICS- B S GREWAL