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AS-404

DISCRETE MATHEMATIS AND ITS APPLICATION

ASSIGNMENT

Solved Problems

PROPOSITIONS AND TRUTH TABLES

4.1. Let p be "It is cold" and let q be "It is raining". Give a simple verbal sentence which describes each of the following statements: (*a*) $\neg p$; (*b*) $p \wedge q$; (*c*) $p \vee q$; (*d*) $q \vee \neg p$.

In each case, translate $\wedge$, $\vee$, and $\sim$ to read "and," "or," and "It is false that" or "not," respectively, and then simplify the English sentence.

(*a*) It is not cold. (*c*) It is cold or it is raining.

(*b*) It is cold and raining. (*d*) It is raining or it is not cold.

4.2. Find the truth table of $\neg p \wedge q$.

Construct the truth table of $\neg p \wedge q$ as in Fig. 4-9(*a*).

p	q	$\neg p$	$\neg p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

(*a*) $\neg p \wedge q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

(*b*) $p \vee \neg(p \wedge q)$

Fig. 4-9

4.3. Verify that the proposition $p \vee \neg(p \wedge q)$ is a tautology.

Construct the truth table of $p \vee \neg(p \wedge q)$ as shown in Fig. 4-9(*b*). Since the truth value of $p \vee \neg(p \wedge q)$ is T for all values of p and q, the proposition is a tautology.

4.4. Show that the propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Construct the truth tables for $\neg(p \wedge q)$ and $\neg p \vee \neg q$ as in Fig. 4-10. Since the truth tables are the same (both propositions are false in the first case and true in the other three cases), the propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent and we can write

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(*a*) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(*b*) $\neg p \vee \neg q$

Fig. 4-10

4.5. Use the laws in Table 4-1 to show that $\neg(p \wedge q) \vee (\neg p \wedge q) \equiv \neg p$.

	Statement	**Reason**
(1)	$\neg(p \vee q) \vee (\neg p \wedge q) \equiv (\neg p \wedge \neg q) \vee (\neg p \wedge q)$	DeMorgan's law
(2)	$\equiv \neg p \wedge (\neg q \vee q)$	Distributive law
(3)	$\equiv \neg p \wedge T$	Complement law
(4)	$\equiv \neg p$	Identity law

CONDITIONAL STATEMENTS

4.6. Rewrite the following statements without using the conditional:

(*a*) If it is cold, he wears a hat.

(*b*) If productivity increases, then wages rise.

Recall that "If p then q" is equivalent to "Not p or q;" that is, $p \to q \equiv \neg p \vee q$. Hence,

(*a*) It is not cold or he wears a hat.

(*b*) Productivity does not increase or wages rise.

4.7. Consider the conditional proposition $p \to q$. The simple propositions $q \to p$, $\neg p \to \neg q$ and $\neg q \to \neg p$ are called, respectively, the *converse*, *inverse*, and *contrapositive* of the conditional $p \to q$. Which if any of these propositions are logically equivalent to $p \to q$?

Construct their truth tables as in Fig. 4-11. Only the contrapositive $\neg q \to \neg p$ is logically equivalent to the original conditional proposition $p \to q$.

p	q	$\neg p$	$\neg q$	Conditional $p \to q$	Converse $q \to p$	Inverse $\neg p \to \neg q$	Contrapositive $\neg q \to \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Fig. 4-11

4.8. Determine the contrapositive of each statement:

(*a*) If Erik is a poet, then he is poor.

(*b*) Only if Marc studies will he pass the test.

(*a*) The contrapositive of $p \to q$ is $\neg q \to \neg p$. Hence the contrapositive follows:

If Erik is not poor, then he is not a poet.

(*b*) The statement is equivalent to: "If Marc passes the test, then he studied." Thus its contrapositive is:

If Marc does not study, then he will not pass the test.

4.9. Write the negation of each statement as simply as possible:

(*a*) If she works, she will earn money.

(*b*) He swims if and only if the water is warm.

(*c*) If it snows, then they do not drive the car.

(*a*) Note that $\neg(p \to q) \equiv p \wedge \neg q$; hence the negation of the statement is:

She works or she will not earn money.

(*b*) Note that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q \equiv \neg p \leftrightarrow q$; hence the negation of the statement is either of the following:

He swims if and only if the water is not warm.
He does not swim if and only if the water is warm.

(*c*) Note that $\neg(p \rightarrow \neg q) \equiv p \wedge \neg\neg q \equiv p \wedge q$. Hence the negation of the statement is:

It snows and they drive the car.

ARGUMENTS

4.10. Show that the following argument is a fallacy: $p \rightarrow q, \neg p \vdash \neg q$.

Construct the truth table for $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ as in Fig. 4-12. Since the proposition $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ is not a tautology, the argument is a fallacy. Equivalently, the argument is a fallacy since in the third line of the truth table $p \rightarrow q$ and $\neg p$ are true but $\neg q$ is false.

p	q	$p \rightarrow q$	$\neg p$	$(p \rightarrow q) \wedge \neg p$	$\neg q$	$[(p \rightarrow q) \wedge \neg p] \rightarrow \neg p$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Fig. 4-12

4.11. Determine the validity of the following argument: $p \rightarrow q, \neg p \vdash \neg p$.

Construct the truth table for $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$ as in Fig. 4-13. Since the proposition $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$ is a tautology, the argument is valid.

p	q	$[(p$	$\rightarrow$	$q)$	$\wedge$	$\neg$	$q]$	$\rightarrow$	$\neg$	p
T	T	T	T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	F	T	F	T
F	T	F	T	T	F	F	T	T	T	F
F	F	F	T	F	T	T	F	T	T	F
Step		1	2	1	3	2	1	4	2	1

Fig. 4-13

4.12. Prove the following argument is valid: $p \rightarrow \neg q,\ r \rightarrow q,\ r \vdash \neg p$.

Construct the truth table of the premises and conclusions as in Fig. 4-14(*a*). Now, $p \rightarrow \neg q$, $r \rightarrow q$, and r are true simultaneously only in the fifth row of the table, where $\neg p$ is also true. Hence the argument is valid.

	p	q	r	$p \rightarrow \neg q$	$r \rightarrow q$	$\neg q$
1	T	T	T	F	T	F
2	T	T	F	F	T	F
3	T	F	T	T	F	F
4	T	F	F	T	T	F
5	F	T	T	T	T	T
6	F	T	F	T	T	T
7	F	F	T	T	F	T
8	F	F	F	T	T	T

(*a*)

p	q	$\neg q$	$p \rightarrow \neg q$	$\neg p$
T	T	F	F	F
T	F	T	T	F
F	T	F	T	T
F	F	T	T	T

(*b*)

Fig. 4-14

4.13. Determine the validity of the following argument:

If 7 is less than 4, then 7 is not a prime number.
7 is not less than 4.

---

7 is a prime number.

First translate the argument into symbolic form. Let p be "7 is less than 4" and q be "7 is a prime number." Then the argument is of the form

$$p \to \neg q, \ \neg q \vdash q$$

Now, we construct a truth table as shown in Fig. 4-14(*b*). The above argument is shown to be a fallacy since, in the fourth line of the truth table, the premises $p \to \neg q$ and $\neg p$ are true, but the conclusion q is false.

Remark: The fact that the conclusion of the argument happens to be a true statement is irrelevant to the fact that the argument presented is a fallacy.

4.14. Test the validity of the following argument:

If two sides of a triangle are equal, then the opposite angles are equal.
Two sides of a triangle are not equal.

The opposite angles are not equal.

First translate the argument into the symbolic form $p \to q, \neg p \vdash \neg q$, where p is "Two sides of a triangle are equal" and q is "The opposite angles are equal." By Problem 4.10, this argument is a fallacy.

Remark: Although the conclusion *does* follow from the second premise and axioms of Euclidean geometry, the above argument does not constitute such a proof since the argument is a fallacy.

QUANTIFIERS AND PROPOSITIONAL FUNCTIONS

4.15. Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of each of the following statements:

(*a*) $(\exists x \in A)(x + 3 = 10)$ (*c*) $(\exists x \in A)(x + 3 < 5)$
(*b*) $(\forall x \in A)(x + 3 < 10)$ (*d*) $(\forall x \in A)(x + 3 \leq 7)$

(*a*) False. For no number in A is a solution to $x + 3 = 10$.

(*b*) True. For every number in A satisfies $x + 3 < 10$.

(*c*) True. For if $x_0 = 1$, then $x_0 + 3 < 5$, i.e., 1 is a solution.

(*d*) False. For if $x_0 = 5$, then $x_0 + 3$ is not less than or equal 7. In other words, 5 is not a solution to the given condition.

4.16. Determine the truth value of each of the following statements where $\mathbf{U} = \{1, 2, 3\}$ is the universal set: (*a*) $\exists x \forall y, x^2 < y + 1$; (*b*) $\forall x \exists y, x^2 + y^2 < 12$; (*c*) $\forall x \forall y, x^2 + y^2 < 12$.

(*a*) True. For if $x = 1$, then 1, 2, and 3 are all solutions to $1 < y + 1$.

(*b*) True. For each x_0, let $y = 1$; then $x_0^2 + 1 < 12$ is a true statement.

(*c*) False. For if $x_0 = 2$ and $y_0 = 3$, then $x_0^2 + y_0^2 < 12$ is not a true statement.

4.17. Negate each of the following statements:

$$(a)\ \exists x\ \forall y,\ p(x, y); \quad (b)\ \exists x\ \forall y,\ p(x, y); \quad (c)\ \exists y\ \exists x\ \forall z,\ p(x, y, z).$$

Use $\neg \forall x\, p(x) \equiv \exists x \neg p(x)$ and $\neg \exists x\, p(x) \equiv \forall x \neg p(x)$:

(*a*) $\neg(\exists x \forall y,\ p(x, y)) \equiv \forall x \exists y \neg p(x, y)$

(*b*) $\neg(\forall x \forall y,\ p(x, y)) \equiv \exists x \exists y \neg p(x, y)$

(*c*) $\neg(\exists y\ \exists x\ \forall z,\ p(x, y, z)) \equiv \forall y\ \forall x\ \exists z \neg p(x, y, z)$

4.18. Let $p(x)$ denote the sentence "$x + 2 > 5$." State whether or not $p(x)$ is a propositional function on each of the following sets: (*a*) N, the set of positive integers; (*b*) $M = \{-1, -2, -3, \ldots\}$; (*c*) C, the set of complex numbers.

(*a*) Yes.

(*b*) Although $p(x)$ is false for every element in M, $p(x)$ is still a propositional function on M.

(*c*) No. Note that $2i + 2 > 5$ does not have any meaning. In other words, inequalities are not defined for complex numbers.

4.19. Negate each of the following statements: (*a*) All students live in the dormitories. (*b*) All mathematics majors are males. (*c*) Some students are 25 years old or older.

Use Theorem 4.4 to negate the quantifiers.

(*a*) At least one student does not live in the dormitories. (Some students do not live in the dormitories.)

(*b*) At least one mathematics major is female. (Some mathematics majors are female.)

(*c*) None of the students is 25 years old or older. (All the students are under 25.)

Supplementary Problems

PROPOSITIONS AND TRUTH TABLES

4.20. Let p denote "He is rich" and let q denote "He is happy." Write each statement in symbolic form using p and q. Note that "He is poor" and "He is unhappy" are equivalent to $\neg p$ and $\neg q$, respectively.

(a) If he is rich, then he is unhappy.
(b) He is neither rich nor happy.
(c) It is necessary to be poor in order to be happy.
(d) To be poor is to be unhappy.

4.21. Find the truth tables for. (a) $p \vee \neg q$; (b) $\neg p \wedge \neg q$.

4.22. Verify that the proposition $(p \wedge q) \wedge \neg(p \vee q)$ is a contradiction.

ARGUMENTS

4.23. Test the validity of each argument:

(a) If it rains, Erik will be sick.
It did not rain.
____
Erik was not sick.

(b) If it rains, Erik will be sick.
Erik was not sick.
____
It did not rain.

4.24. Test the validity of the following argument:

If I study, then I will not fail mathematics.
If I do not play basketball, then I will study.
But I failed mathematics.
____
Therefore I must have played basketball.

QUANTIFIERS

4.25. Let $A = \{1, 2, \ldots, 9, 10\}$. Consider each of the following sentences. If it is a statement, then determine its truth value. If it is a propositional function, determine its truth set.

(a) $(\forall x \in A)(\exists y \in A)(x + y < 14)$
(b) $(\forall y \in A)(x + y < 14)$
(c) $(\forall x \in A)(\forall y \in A)(x + y < 14)$
(d) $(\exists y \in A)(x + y < 14)$

4.26. Negate each of the following statements:

(a) If the teacher is absent, then some students do not complete their homework.

(b) All the students completed their homework and the teacher is present.

(c) Some of the students did not complete their homework or the teacher is absent.

4.27. Negate each statement in Problem 4.15.

4.28. Find a counterexample for each statement were $\mathbf{U} = \{3, 5, 7, 9\}$ is the universal set:

(a) $\forall x, x + 3 \geq 7$, (b) $\forall x, x$ is odd, (c) $\forall x, x$ is prime, (d) $\forall x, |x| = x$

Answers to Supplementary Problems

4.20. (a) $p \to \neg q$; (b) $\neg p \wedge \neg q$; (c) $q \to \neg p$; (d) $\neg p \to \neg q$.

4.21. (a) T, T, F, T; (b) F, F, F, T.

4.22. Construct its truth table. It is a contradiction since its truth table is false for all values of p and q.

4.23. First translate the arguments into symbolic form: p for "It rains," and q for "Erik is sick:"

(a) $p \to q, \neg p \vdash \neg q$ (b) $p \to q, \neg q \vdash \neg p$

By Problem 4.10, (a) is a fallacy. By Problem 4.11, (b) is valid.

4.24. Let p be "I study," q be "I failed mathematics," and r be "I play basketball." The argument has the form:

$$p \to \neg q, \neg r \to p, q \vdash r$$

Construct the truth tables as in Fig. 4-15, where the premises $p \to \neg q, \neg r \to p$, and q are true simultaneously only in the fifth line of the table, and in that case the conclusion r is also true. Hence the argument is valid.

p	q	r	$\neg q$	$p \to \neg q$	$\neg r$	$\neg r \to p$
T	T	T	F	F	F	T
T	T	F	F	F	T	T
T	F	T	T	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	T	F	F	T	T	F
F	F	T	T	T	F	T
F	F	F	T	T	T	F

Fig. 4-15

4.25. (a) The open sentence in two variables is preceded by two quantifiers; hence it is a statement. Moreover, the statement is true.

(b) The open sentence is preceded by one quantifier; hence it is a propositional function of the other variable. Note that for every $y \in A$, $x_0 + y < 14$ if and only if $x_0 = 1, 2$, or 3. Hence the truth set is $\{1, 2, 3\}$.

(c) It is a statement and it is false: if $x_0 = 8$ and $y_0 = 9$, then $x_0 + y_0 < 14$ is not true.

(d) It is an open sentence in x. The truth set is A itself.

4.26. (a) The teacher is absent and all the students completed their homework.

(b) Some of the students did not complete their homework or the teacher is absent.

(c) All the students completed their homework and the teacher is present.

4.27. (a) $(\forall x \in A)(x + 3 \neq 10)$ (c) $(\forall x \in A)(x + 3 \geq 5)$

(b) $(\exists x \in A)(x + 3 \geq 10)$ (d) $(\exists x \in A)(x + 3 > 7)$

4.28. (a) Here 3 is a counterexample.

(b) The statement is true; hence no counterexample exists.

(c) Here 9 is the only counterexample.

(d) The statement is true; hence there is no counterexample.

1.8 MATHEMATICAL INDUCTION

An essential property of the set $\mathbf{N} = \{1, 2, 3, \ldots\}$ of positive integers follows:

Principle of Mathematical Induction I: Let P be a proposition defined on the positive integers $\mathbf{N}$; that is, $P(n)$ is either true or false for each $n \in \mathbf{N}$. Suppose P has the following two properties:

(i) $P(1)$ is true.

(ii) $P(k + 1)$ is true whenever $P(k)$ is true.

Then P is true for every positive integer $n \in \mathbf{N}$.

We shall not prove this principle. In fact, this principle is usually given as one of the axioms when $\mathbf{N}$ is developed axiomatically.

EXAMPLE 1.13 Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n) : 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

(The kth odd number is $2k - 1$, and the next odd number is $2k + 1$.) Observe that $P(n)$ is true for $n = 1$; namely,

$$P(1) = 1^2$$

Assuming $P(k)$ is true, we add $2k + 1$ to both sides of $P(k)$, obtaining

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) - k^2 + (2k + 1) = (k + 1)^2$$

which is $P(k + 1)$. In other words, $P(k + 1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction, P is true for all n.

There is a form of the principle of mathematical induction which is sometimes more convenient to use. Although it appears different, it is really equivalent to the above principle of induction.

Principle of Mathematical Induction II: Let P be a proposition defined on the positive integers $\mathbf{N}$ such that:

(i) $P(1)$ is true.

(ii) $P(k)$ is true whenever $P(j)$ is true for all $1 \le j < k$.

Then P is true for every positive integer $n \in \mathbf{N}$.

Remark: Sometimes one wants to prove that a proposition P is true for the set of integers

$$\{a, a + 1, a + 2, a + 3, \ldots\}$$

where a is any integer, possibly zero. This can be done by simply replacing 1 by a in either of the above Principles of Mathematical Induction.

MATHEMATICAL INDUCTION

1.24 Prove the proposition $P(n)$ that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$; that is,

$$P(n) = 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

The proposition holds for $n = 1$ since:

$$P(1): 1 = \tfrac{1}{2}(1)(1+1)$$

Assuming $P(k)$ is true, we add $k+1$ to both sides of $P(k)$, obtaining

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \tfrac{1}{2}k(k+1) + (k+1) \\ &= \tfrac{1}{2}[k(k+1) + 2(k+1)] \\ &= \tfrac{1}{2}[(k+1)(k+2)] \end{aligned}$$

which is $P(k+1)$. That is, $P(k+1)$ is true whenever $P(k)$ is true. By the Principle of Induction, P is true for all n.

1.25 Prove the following proposition (for $n \geq 0$):

$$P(n): 1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$$

$P(0)$ is true since $1 = 2^1 - 1$. Assuming $P(k)$ is true, we add 2^{k+1} to both sides of $P(k)$, obtaining

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2(2^{k+1}) - 1 = 2^{k+2} - 1$$

which is $P(k+1)$. That is, $P(k+1)$ is true whenever $P(k)$ is true. By the principle of induction, $P(n)$ is true for all n.

INDUCTION

1.50 Prove: $2 + 4 + 6 + \cdots + 2n = n(n+1)$

1.51 Prove: $1 + 4 + 7 + \cdots + 3n - 2 = \frac{n(3n-1)}{2}$

1.52 Prove: $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

1.53 Prove: $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

1.54 Prove: $\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$

1.55 Prove $7^n - 2^n$ is divisible by 5 for all $n \in \mathbf{N}$

1.56 Prove $n^3 - 4n + 6$ is divisible by 3 for all $n \in \mathbf{N}$

1.57 Use the identity $1 + 2 + 3 + \cdots + n = n(n+1)/2$ to prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$