Differential equation of SVF with increasing discharg

Assumption →

1→ The pressure distribution is assumed to be hydrostatic.

2→ The one-dimensional method of analysis is adopted.

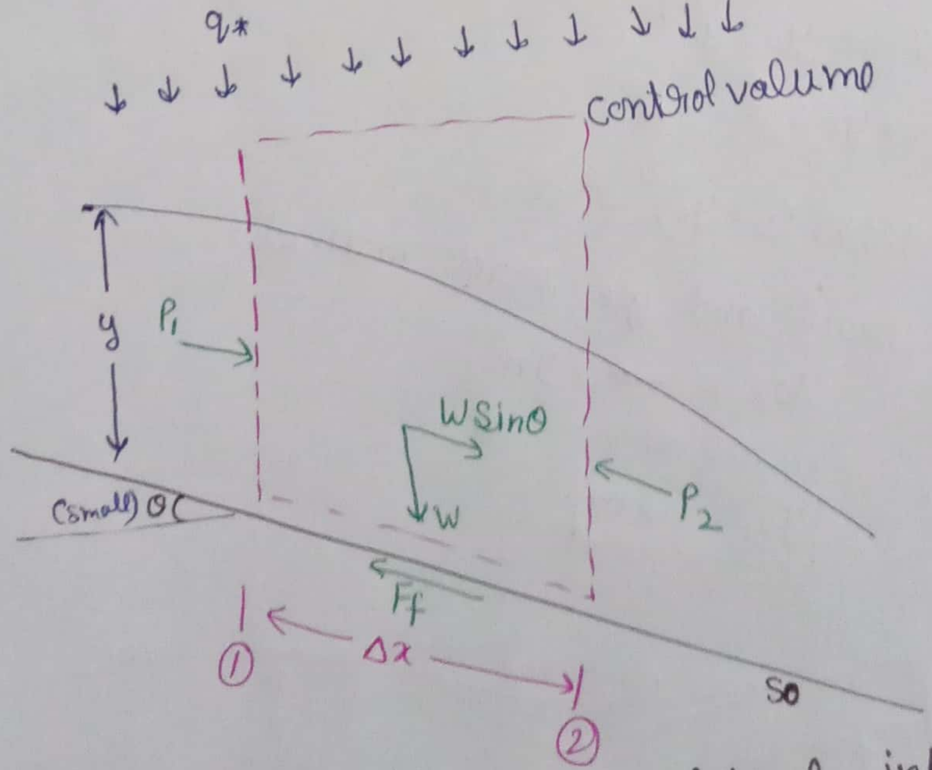
3→ The frictional losses in SVF are assumed to be adequately represented by a uniform flow resistance equation, such as manning's formula.

4→ The effect of air entrainment on forces involved in the momentum equation is neglected.

5→ It is assumed that the lateral flow does not contribute any momentum in the longitudinal direction.

6→ The flow is considered to be steady.

7→ The channel is prismatic and is of small slope.



Definition sketch of SVF with lateral inflow

Consider a control volume formed by two sections 1 and 2, distance Δx apart.

M = momentum flux = $\frac{\beta \rho Q^2}{A}$

P = pressure force = $\gamma A \bar{y}$

F_f = frictional force = $\gamma A S_f \Delta x$

$W \sin\theta$ = Component of the weight of the control volume in x direction.

$\bar{y}$ = depth of the centre of gravity of the flow cross-section from the water-surface.

$W = \gamma A . \Delta x$

A = cross sectional area of control volume.

γ = unit weight of water.

ρ = mass density of water.

$\boxed{\gamma = \rho g}$

S_f = Energy slope.

S_o = bed slope.

It is difficult to assess the net energy imparted to the flow and as such the energy equation is not of much use in developing the equation of motion.

Applying the momentum equation in the longitudinal x direction –

$$M_2 - M_1 = P_1 - P_2 + W \sin\theta - F_f$$

$$\Delta M = -\Delta P + W \sin\theta - F_f$$

dividing by Δx both side

$$\frac{\Delta M}{\Delta x} = -\frac{\Delta P}{\Delta x} + \frac{W \sin\theta}{\Delta x} - \frac{F_f}{\Delta x}$$

$$\frac{\Delta M}{\Delta x} = -\frac{\Delta P}{\Delta x} + \frac{\gamma A . \Delta x . S_o}{\Delta x} - \frac{\gamma A S_f \Delta x}{\Delta x}$$

$$\frac{dM}{dx} = -\frac{dP}{dx} + \gamma A S_o - \gamma A S_f \quad \text{——— (1)}$$

$$\frac{dM}{dx} = \frac{d}{dx}\left(\beta \rho \frac{Q^2}{A}\right) \qquad \because Q, A \text{ are dependent on } x$$

$$= \beta\rho\left(\frac{d}{dQ}\frac{Q^2}{A}\cdot\frac{dQ}{dx} + \frac{d}{dA}\frac{Q^2}{A}\frac{dA}{dx}\right)$$

$$= \beta\rho\left(\frac{2Q}{A}\frac{dQ}{dx} + \frac{(-1)Q^2}{A^2}\frac{dA}{dx}\right)$$

$$= \rho\beta\left(\frac{2Q}{A}\cdot q_* - \frac{Q^2}{A^2}\frac{dA}{dy}\cdot\frac{dy}{dx}\right) \qquad \because q_* = \frac{dQ}{dx}$$

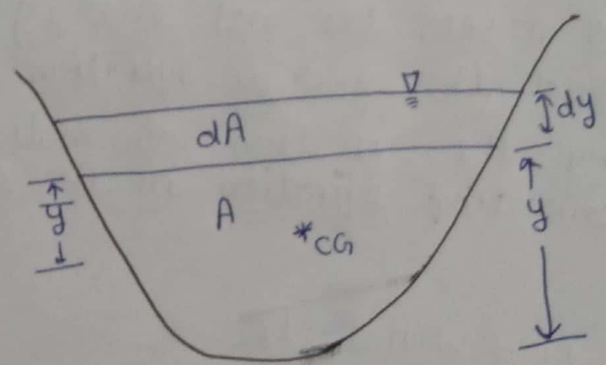
$$\because \frac{dA}{dy} = T$$

$$\frac{dM}{dx} = \rho\beta\left(\frac{2Q}{A}q_* - \frac{Q^2 T}{A^2}\frac{dy}{dx}\right) \quad \text{———— (2)}$$

$$\frac{dP}{dx} = \frac{d}{dx}(\gamma A \bar{y})$$

$$= \gamma\left[\frac{d}{dx}A\bar{y}\right] = \gamma\left[\cancel{\frac{d}{dA}A\bar{y}\cdot\frac{dA}{dx} + \frac{d}{dy}A\bar{y}\cdot\frac{dy}{dx}}\right].$$

$$\frac{dP}{dx} = \gamma\left[A\frac{d\bar{y}}{dx} + \bar{y}\frac{dA}{dx}\right] \quad \text{———— (3)}$$



By taking moments of the areas about the new water surface after a small change dy in depth.

$$A(\bar{y} + dy) + dA\cdot\frac{dy}{2} = (A + dA)(\bar{y} + d\bar{y})$$

$$A\bar{y} + A\,dy + \frac{dA\cdot dy}{2} = A\bar{y} + A\cdot d\bar{y} + \bar{y}\cdot dA + dA\cdot d\bar{y}$$

By neglecting second-order small quantities,

$$A\,d\bar{y} + \bar{y}\,dA = A.dy$$

dividing by dx both sides

$$A\frac{d\bar{y}}{dx} + \bar{y}\frac{dA}{dx} = A\frac{dy}{dx}$$

from eq (3)

$$\frac{dP}{dx} = \gamma A\frac{dy}{dx} \quad \text{——— (4)}$$

eq (1) $\rightarrow$ $\frac{dM}{dx} = -\frac{dP}{dx} + \gamma A S_o - \gamma A S_f$

from eq (2)

$$\rho\beta\left[\frac{2Q}{A}q_* - \frac{Q^2 T}{A^2}\frac{dy}{dx}\right] = -\gamma A\frac{dy}{dx} + \gamma A S_o - \gamma A S_f \qquad \because \rho = \frac{\gamma}{g}$$

$$\frac{\gamma\beta}{g}\cdot\frac{2Qq_*}{A} - \frac{\gamma\beta}{g}\cdot\frac{Q^2 T}{A^2}\frac{dy}{dx} = -\gamma A\frac{dy}{dx} + \gamma A S_o - \gamma A S_f$$

By dividing (γA) both side.

$$\frac{2\beta Q q_*}{gA^2} - \frac{\beta Q^2 T}{gA^3}\frac{dy}{dx} = -\frac{dy}{dx} + S_o - S_f$$

$$\frac{dy}{dx} - \frac{\beta Q^2 T}{gA^3}\frac{dy}{dx} = S_o - S_f - \frac{2\beta Q q_*}{gA^2}$$

$$\frac{dy}{dx}\left[1 - \frac{\beta Q^2 T}{gA^3}\right] = S_o - S_f - \frac{2\beta Q q_*}{gA^2}$$

$$\boxed{\frac{dy}{dx} = \frac{S_o - S_f - \frac{2\beta Q q_*}{gA^2}}{1 - \frac{\beta Q^2 T}{gA^3}}}$$