Unit - 5

Application of Partial differential equation

Q1 Using the method of separation of variables, Solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, \text{ where } u(x,0) = 6e^{-3x}.$$

Sol.

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{—— (1)}$$

Let $u = X(x)T(t)$ be C.S of eq (1) —— (2)

$$\frac{\partial u}{\partial x} = X'T \quad , \quad \frac{\partial u}{\partial t} = XT'$$

Putting the value of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ & u in eq (1) we get

$$X'T = 2XT' + XT$$

Separating the variables, we get

$$\frac{X' - X}{2X} = \frac{T'}{T} = -p^2$$

From 1st term and last term, we get

$$\frac{X' - X}{2X} = -p^2$$

$$X' - X + 2p^2 X = 0$$

$$(D - (1+2p^2))X = 0$$

AE $\quad m - (1+2p^2) = 0$

$$m = (1+2p^2)$$

C.F $\quad c_1 e^{(1+2p^2)x}$

P.I $= 0$

$$\boxed{X = c_1 e^{(1+2p^2)x}}$$

taking last two

$$\frac{T'}{T} = -p^2 \qquad \left[T' = \frac{dT}{dt}\right]$$

$$\log T = -p^2 t + \log c_2$$

$$\boxed{T = c_2 e^{-p^2 t}}$$

Substituting the value of X and T in eq (2) we get

$$u = c_1 e^{(1+2p^2)x} . c_2 e^{-p^2 t} = c_1 c_2 e^{(1+2p^2)x - p^2 t} \quad \text{—— (3)}$$

Using $u(x,0) = 6e^{-3x}$ in (3) we get

$$6e^{-3x} = c_1 c_2 e^{(1+2p^2)x}$$

$c_1 c_2 = 6 \quad , \quad 1+2p^2 = -3 \Rightarrow 2p^2 = -4$, $p^2 = -2$ $\quad | \quad u(x,t) = 6e^{-3x+2t}$

One dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

B.C. (i) $u(0,t)=0$

(ii) $u(l,t)=0$

(iii) $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$ (iv) $u(x,0)=f(x)$

Q. A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $u=u_0\sin^3\left(\frac{\pi x}{l}\right)$. If it is released from the rest from this position, find the displacement $u(x,t)$.

Sol. We know that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ — (1)

Let $u = X(x)T(t)$ be C.S of eq (1) — (2)

$\frac{\partial^2 u}{\partial x^2} = X''T$ & $\frac{\partial^2 u}{\partial t^2} = XT''$

Substituting these value in eq (1) we get

$$X''T = \frac{1}{c^2}XT''$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -p^2 \text{ (say)}$$

Solving first and last we get

$\frac{X''}{X} = -p^2 \Rightarrow (D^2+p^2)X=0$

A.E $m^2+p^2=0$

$m=\pm ip$

C.F $c_1\cos px + c_2\sin px$

P.I $=0$

$$\boxed{X = c_1\cos px + c_2\sin px}$$ — (3)

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Taking last two, we get

$$\frac{1}{c^2}\frac{T''}{T} = -p^2 \Rightarrow T'' + p^2c^2T = 0$$

$$(D^2 + p^2c^2)T = 0$$

AE $\quad m^2 + p^2c^2 = 0 \Rightarrow m = ipc$

CF $\quad c_3 \cos pct + c_4 \sin pct$

$PI = 0$

$$\boxed{T = c_3 \cos pct + c_4 \sin pct}$$

Putting the value of x and T in eq in (2) we get

$$u = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct) \quad \text{—} (4)$$

using $u(0,t) = 0$ in eq (4) we get

$$0 = c_1 (c_3 \cos pct + c_4 \sin pct)$$

$\boxed{c_1 = 0}$ putting in (4)

$$u = c_2 \sin px \,(c_3 \cos pct + c_4 \sin pct) \quad \text{—} (5)$$

using $(l,t) = 0$ in eq (5) we get

$$0 = c_2 \sin pl \,(c_3 \cos pct + c_4 \sin pct)$$

$$\sin pl = 0 = \sin n\pi$$

$$pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}} \text{ putting in (5)}$$

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right) \quad \text{—} (6)$$

Differentiating w.r.t 't' we get

$$\frac{\partial u}{\partial t} = c_2 \sin \frac{n\pi x}{l} \left[c_3 \left(-\sin \frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right) + c_4 \cos \frac{n\pi ct}{l}\left(\frac{n\pi c}{l}\right)\right] \quad \text{—} (7)$$

using $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$ in eq (7) we get

$$0 = c_2 \sin \frac{n\pi x}{l} \cdot c_4 \left(\frac{n\pi c}{l}\right)$$

$\boxed{c_4 = 0}$ putting in (6) we get

$$u = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \text{—} (8) \qquad [c_2 c_3 = b_n]$$

$$u = c_1 c_3 \sin\frac{n\pi x}{l} \cos\frac{n\pi c t}{l} \quad \text{---(5)} \qquad [c_1 c_3 = b_n]$$

$$u = b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c t}{l} \quad \text{---(6)}$$

using $u(x,0) = u_0 \sin^3\left(\frac{\pi x}{l}\right)$ in eq(6) we get

$$u_0 \sin^3\frac{\pi x}{l} = b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c 0}{l}$$

$$\frac{u_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = b_1 \sin\frac{\pi x}{l}\cos\frac{\pi c 0}{l} + b_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi c 0}{l} + b_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi c 0}{l}$$

$$\frac{u_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l}$$

on comparing

$b_1 = \frac{3u_0}{4}$, $b_2 = 0$, $b_3 = -\frac{u_0}{4}$ putting in (5) we get

$$u = b_1 \sin\frac{\pi x}{l}\cos\frac{\pi c t}{l} + b_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi c t}{l}$$

$$u = \frac{u_0}{4}\left[3\sin\frac{\pi x}{l}\cos\frac{\pi c t}{l} - \sin\frac{3\pi x}{l}\cos\frac{3\pi c t}{l}\right] \quad \text{Ans}$$

One dimensional heat flow

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

B.C.

(i) $u(0,t) = 0$

(ii) $u(l,t) = 0$

(iii) $u(x,0) =$ [given in Problem]

Q: A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°c and are kept at that temperature. Prove that the temperature function $u(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{\pi^2 c^2 n^2 t}{l^2}}$$

where b_n is determined from the equation $U_0 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Sol. We know that heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

Let $u(x,t) = X(x)T(t)$ —— (2) be CS of eq (1)

$$\frac{\partial u}{\partial t} = XT' \quad \& \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting value of $\frac{\partial u}{\partial t}$ & $\frac{\partial^2 u}{\partial x^2}$ in eq (1) we get

$$XT' = c^2 X''T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = -p^2 \text{ (say)}$$

taking first and last we get

$$X'' + p^2 X = 0$$

$$(D^2 + p^2)X = 0$$

AE $m^2 + p^2 = 0 \Rightarrow m = \pm ip$

C.F $c_1 \cos px + c_2 \sin px$

$PI = 0$

$$\boxed{X = c_1 \cos px + c_2 \sin px}$$

taking last two

$$\frac{T'}{T} = -p^2c^2$$

$$\log T = -p^2c^2t + \log c_3$$

$$T = c_3 e^{-p^2c^2t}$$

Putting the value of X and T in eq (2) we get

$$u = (c_1 \cos px + c_2 \sin px) c_3 e^{-p^2c^2t} \quad \text{—— (3)}$$

$$u = (c_1 \cos px + c_2 \sin px) c_3 e^{-p^2 c^2 t} \quad \text{--- (3)}$$

using $u(0,t) = 0$ in eq (3) we get

$$0 = c_1 c_3 e^{-p^2 c^2 t}$$

$\boxed{c_1 = 0}$ putting in (3)

$$u = c_2 c_3 \sin px \, e^{-p^2 c^2 t} \quad \text{--- (4)}$$

using $u(l,t) = 0$ in eq (4) we get

$$0 = b_n \sin pl \, e^{-p^2 c^2 t} \qquad [c_2 c_3 = b_n]$$

$$\sin pl = 0 = \sin n\pi$$

$\boxed{p = \frac{n\pi}{l}}$ putting in (4)

$$u = b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c^2 t}{l}} \quad \text{--- (5)}$$

using $u(x,0) = U_0$ in eq (5) we get

$$\boxed{U_0 = b_n \sin \frac{n\pi x}{l}}$$

Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

B.C. (i) $u(0,y) = 0$ (ii) $u(l,y) = 0$

(iii) $u(x,0) = 0$ (iv) $u(x,a) = \sin\left(\frac{p\pi x}{l}\right)$