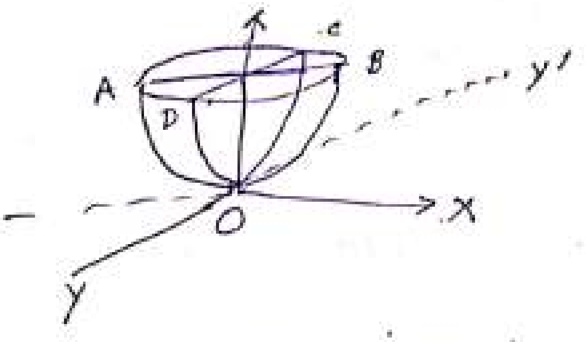
Paraboloids

Equation of elliptic paraboloid



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$$

The sections by planes parallel to $x=0$ are given by

$$\frac{y^2}{b^2} = \frac{2z}{c} - \frac{k^2}{a^2}, \quad x = k$$

Similarly we can write the eqs of sections by planes $\parallel$ to $y=0$ and ~~z=0~~.

i.e plane section of paraboloid by planes $\parallel$ to yz and zx - planes are paraboloids.

The equation of Hyperbolic Paraboloid -

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$$

The sections by the planes $z=k$ are the similar hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2k}{c}$, $z=k$.

whose transverse and conjugate axes are parallel to x and y axes and increase with k, $k>0$.

The sections by the plane $z=0$ is the pair of lines

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad z=0 \quad \text{i.e } \frac{x}{a} = \frac{y}{b}, z=0, \frac{x}{a} = -\frac{y}{b}, z=0$$

The sections by the planes parallel to yz and zx planes are parabolas.

General equation of paraboloid

$$ax^2 + by^2 = 2cz$$

It is elliptic paraboloid if a and b both are either positive or negative.

For ~~all~~ hyperbolic paraboloid, a and b must be of opposite signs.

Diameters of a paraboloid –

If a line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ meets the paraboloid $ax^2 + by^2 = 2cz$ we get

$$r^2(al^2 + bm^2) + 2r(a\alpha l + b\beta m - cn) + (a\alpha^2 + b\beta^2 - 2c\gamma) = 0$$

ie lines cut the paraboloid in two points and plane sections of the paraboloids are conics

Now if $l = 0$, $m = 0$, then one value of r is ∞ hence any line $\parallel$ to z-axis meets the paraboloids in one point only.

Such lines are called Diameters of paraboloid.

1. Equation of tangent plane of Paraboloid is

$$a\alpha x + b\beta y = 2c(z+\gamma)$$

2. Condition of tangency $\frac{l^2}{a} + \frac{m^2}{b} + \frac{2np}{c} = 0$

when equation of the plane is $lx + my + nz = p$.

3. The plane

$$2n(lx+my+nz)+c\left(\frac{l^2}{a}+\frac{m^2}{b}\right)=0$$

touches the paraboloid.

4. Point of contact is $\left(-\frac{lc}{an}, -\frac{mc}{bn}, -\frac{p}{n}\right)$

5. The locus of point of intersection of three mutually perpendicular tangent planes ~~are~~ of the paraboloid -

$$2z+c\left(\frac{1}{a}+\frac{1}{b}\right)=0.$$

which is the plane at right angles to the z-axis, the axis of paraboloid.

6. Equation of normal at (α, β, γ) is

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{-c}.$$

7. Equation of polar plane is $a\alpha x+b\beta y=c(z+\gamma)$

8. The equation of enveloping cone with (α,β,γ) as its vertex is $SS_1=T^2$

$$(ax^2+by^2-2cz)(a\alpha^2+b\beta^2-2c\gamma)=(a\alpha x+b\beta y-c\gamma-cz)^2$$

9. Equation of enveloping cylinder with generators ∥ to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is

$$(ax^2+by^2-2c\gamma z)(al^2+bm^2)=(alx+bmy-cn)^2$$

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10. Its plane of contact is

$$alx + bmy - cn = 0.$$

11. Locus of chords bisected at (α, β, γ) is the plane

$$a\alpha x + b\beta y - cz = a\alpha^2 + b\beta^2 - c\gamma \quad \text{ie } S_1 = T.$$

which meets the paraboloid in a conic whose centre is (α, β, γ).

12. Locus of mid points of a system of parallel chords with direction ratios l, m, n is the plane

$$alx + bmy - cn = 0.$$

which are parallel to z-axis and is called diametral plane conjugate to given direction.

<u>Ex</u> Show that the plane $2x - 4y - z + 3 = 0$ touches the paraboloid $x^2 - 2y^2 = 3z$ and find the coordinates of the point of contact.

∴ Condition of tangence is $\frac{l^2}{a} + \frac{m^2}{b} + \frac{2np}{c} = 0$

or $$\frac{4}{1} + \frac{16}{-2} + \frac{-2 \times -3}{3/2} = 0.$$

Hence plane touches the paraboloid.

Now tangent plane of the paraboloid is $xx' - 2yy' = \frac{3}{2}(z + z')$

on comparing with given plane we get

$$\frac{x'}{2} = \frac{-2y'}{-4} = \frac{-3/2}{-1} = \frac{-3z'/2}{3}. \quad \text{ie } (3, 3, -3)$$

Numbers of normals that can be drawn through a given point to a paraboloid.

Let $ax^2 + by^2 = 2cz$ be the eq of paraboloid.

If the normal at any point (x', y', z') to paraboloid passes through a point (α, β, γ) then eq of the normal is

$$\frac{\alpha - x'}{ax'} = \frac{\beta - y'}{by'} = \frac{\gamma - z'}{-c} = \lambda \text{ (say)}$$

$$\Rightarrow \quad x' = \frac{\alpha}{1 + a\lambda}, \quad y' = \frac{\beta}{1 + b\lambda}, \quad z' = \gamma + c\lambda$$

But (x', y', z') must lie on $ax^2 + by^2 = 2cz$

$$\therefore \quad ax'^2 + by'^2 = 2cz'$$

Substituting the values of x', y' & z' we get.

$$\frac{a\alpha^2}{(1+a\lambda)^2} + \frac{b\beta^2}{(1+b\lambda)^2} = 2c(\gamma + c\lambda)$$

The equation is fifth degree in λ.

$\Rightarrow$ Five normals can be drawn to a paraboloid so as to pass through a given point (α, β, γ).

<u>example</u> Find the equation of the plane, which cuts the paraboloid $x^2 - 2y^2 = 3z$ in the conic with centre $(1, 2, 3)$

Sol. using the formula $S_1 = T$ i.e. the locus of chords bisected at (α, β, γ) is the plane is

$$a\alpha x + b\beta y - c(\gamma + z) = a\alpha^2 + b\beta^2 - 2c\gamma$$

$$x - 4y - \frac{(9 - 3z)}{2} = 1 - 8 - 9$$

$$2x - 8y - 3z = -32 + 9$$

$$2x - 8y - 3z + 23 = 0$$

Ex Find the equation of normal at $(4, 3, 5)$ on the paraboloid $\frac{x^2}{2} - \frac{y^2}{2} = z$.

Sol. Comparing the eq by $ax^2 + by^2 = 2cz$

$$a = 1/2 \quad , \quad b = -1/2 \quad , \quad c = 1/2$$

or $x^2 - y^2 = 2z$ i.e. $a = 1, b = -1, c = 1$

Using the formula for normal.

$$\frac{x - x'}{ax'} = \frac{y - y'}{by'} = \frac{z - z'}{c}$$

$$\frac{x - 4}{4} = \frac{y - 3}{-3} = \frac{z - 5}{1}$$

is the required eq of normal.