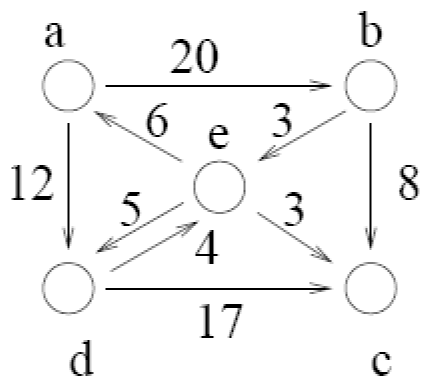
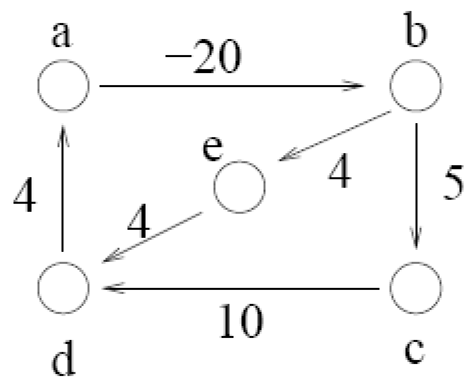
The All-Pairs Shortest Paths Problem

Given a weighted digraph G(V,E) with weight function w: E -> R , (R is the set of real numbers), determine the length of the shortest path (i.e., distance) between all pairs of vertices in G . Here we assume that there are no cycles with zero or negative cost.



Without negative cost cycle



With negative cost cycle

Soution1: If there are no negative cost edges apply Dijkstra's algorithm to each vertex (as the source) of the digraph. Recall the Dijkstra's algorithm run in Θ(V+E(log V)).This gives a **Θ(V(V+E(log V))).**

Input: weighted, directed graph G = (V,E), with weight function w : E ->R.
The weight of path p =< v0, v1……vk > is the sum of the weights of its constituent edges:

$$\mathbf{w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)}$$

The shortest-path weight from u to v is

$$\delta(u,v) = \begin{cases} \min\{w(p)\} & \text{if there is path from u to v} \\ \infty & \text{otherwise} \end{cases}$$

A shortest path from vertex u to vertex v is then defined as any path p with weight w(p) = δ (u,v).

All Pairs Shortest Paths: Compute d(u, v) the shortest path distance from u to v for all pairs of vertices u and v.

Assume that the graph is represented by an n x n matrix with the weights of the edges.

$$W_{ij} = \begin{cases} 0 & \text{if } i{=}{=}j \\ w(i,j) & \text{if } i \neq j \ \&\ (i,j) \in E \\ \infty & \text{if } i \neq j \ \&\ (i,j) \text{ not} \in E \end{cases}$$

Floyd-Warshall, Dynamic Programming

- Let $d^{(k)}ij$ be the weight of a shortest path from vertex i to vertex j for which all intermediate vertices are in the set {1,2…… k}.
- When k = 0, a path from vertex i to vertex j with no intermediate vertex numbered higher than 0 has no intermediate vertices at all, hence $d^{(0)}ij$ = wij.

$$d^{(k)}ij = \begin{cases} wij & \text{if } k=0 \\ \min\{d^{(k-1)}ij,\ d^{(k-1)}ik + d^{(k-1)}kj & \text{if } k >= 1 \end{cases}$$

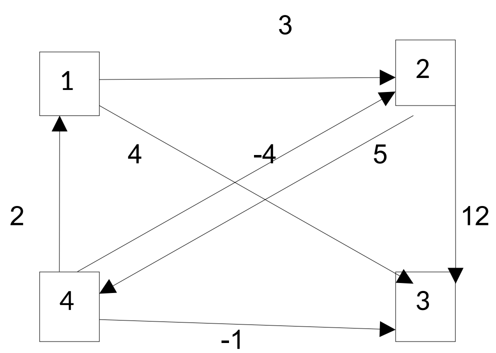
Algorithm

```
Floyd-Warshall(W)
n ← rows[W]
D(0) ← W
for k ← 1 to n
      do for i ← 1 to n
            do for j ← 1 to n
                  do d(k)ij ← min{d(k-1)ij, d(k-1)ik + d(k-1)kj

return D(n)

Running time O(V 3)
```

Example:



D^0

$$\begin{pmatrix} 0 & 3 & \infty & \infty \\ \infty & 0 & 12 & 5 \\ 4 & \infty & 0 & -1 \\ 2 & -4 & \infty & 0 \end{pmatrix}$$

$$D^1 = \begin{pmatrix} 0 & 3 & \infty & \infty \\ \infty & 0 & \mathbf{12} & \mathbf{5} \\ 4 & \mathbf{7} & 0 & \mathbf{-1} \\ 2 & \mathbf{-4} & \infty & 0 \end{pmatrix}$$

$D^1[2,3] = \min(D^0[2,3], D^0[2,1] + D^0[1,3]) = \min(12, \infty+\infty) = 12$

$D^1[2,4] = \min(D^0[2,4], D^0[2,1] + D^0[1,4]) = \min(5, \infty+\infty) = 5$

$D^1[3,2] = \min(D^0[3,2], D^0[3,1] + D^0[1,2]) = \min(\infty, 4+3) = 7$

$D^1[3,4] = \min(D^0[3,4], D^0[3,1] + D^0[1,4]) = \min(-1, 4+\infty) = -1$

$D^1[4,2] = \min(D^0[4,2], D^0[4,1] + D^0[1,2]) = \min(-4, 2+3) = -4$

$D^1[4,3] = \min(D^0[4,3], D^0[4,1] + D^0[1,3]) = \min(\infty, 2+\infty) = \infty$

$$D^2 = \begin{pmatrix} 0 & 3 & \mathbf{15} & \mathbf{8} \\ \infty & 0 & 12 & 5 \\ \mathbf{4} & 7 & 0 & \mathbf{-1} \\ \mathbf{2} & -4 & \mathbf{8} & 0 \end{pmatrix}$$

$D^2[1,3] = \min(D^1[1,3], D^1[1,2] + D^1[2,3]) = \min(\infty, 3+12) = 15$

$D^2[1,4] = \min(D^1[1,4], D^1[1,2] + D^1[2,4]) = \min(\infty, 3+5) = 8$

$D^2[3,1] = \min(D^1[3,1], D^1[3,2] + D^1[2,1]) = \min(4, 7+\infty) = 4$

$D^2[3,4] = \min(D^1[3,4], D^1[3,2] + D^1[2,4]) = \min(-1, 7+5) = -1$

$D^2[4,1] = \min(D^1[4,1], D^1[4,2] + D^1[2,1]) = \min(2, -4+\infty) = 2$

$D^2[4,3] = \min(D^1[4,3], D^1[4,2] + D^1[2,3]) = \min(\infty, -4+12) = 8$

$$D^3 = \begin{pmatrix} 0 & \mathbf{3} & 15 & \mathbf{8} \\ \mathbf{16} & 0 & 12 & \mathbf{5} \\ 4 & 7 & 0 & -1 \\ \mathbf{2} & \mathbf{-4} & 8 & 0 \end{pmatrix}$$

$D^3[1,2] = \min(D^2[1,2], D^2[1,3] + D^2[3,2]) = \min(3, 15+7) = 3$

$D^3[1,4] = \min(D^2[1,4], D^2[1,3] + D^2[3,4]) = \min(8, 15+(-1)) = 8$

$D^3[2,1] = \min(D^2[2,1], D^2[2,3] + D^2[3,1]) = \min(\infty, 12+4) = 16$

$D^3[2,4] = \min(D^2[2,4], D^2[2,3] + D^2[3,4]) = \min(5, 12+(-1)) = 5$

$D^3[4,1] = \min(D^2[4,1], D^2[4,3] + D^2[3,1]) = \min(2, 8+4) = 2$

$D^3[4,2] = \min(D^2[4,2], D^2[4,3] + D^2[3,2]) = \min(-4, 8+7) = -4$

D^4

$$\begin{pmatrix} 0 & \mathbf{3} & \mathbf{15} & 8 \\ \mathbf{7} & 0 & \mathbf{12} & 5 \\ \mathbf{1} & \mathbf{-5} & 0 & -1 \\ 2 & -4 & 8 & 0 \end{pmatrix}$$

D^4 [1,2] = min(D^3 [1,2] , D^3 [1,4] + D^3 [4,2]) = min(3,8+-4) =3

D^4 [1,3] = min(D^3 [1,3] , D^3 [1,4] + D^3 [4,3]) = min(15,8+8) =15

D^4 [2,1] = min(D^3 [2,1] , D^3 [2,4] + D^3 [4,1]) = min(16,5+2) =7

D^4 [2,3] = min(D^3 [2,3] , D^3 [2,4] + D^3 [4,3]) = min(12,5+8) =12

D^4 [3,1] = min(D^3 [3,1] , D^3[3,4] + D^3 [4,1]) = min(4,-1+2) =1

D^4 [3,2] = min(D^3 [3,2] , D^3[3,4] + D^3 [4,2]) = min(7,-1+-4) =-5