Circular Convolution : This is the important porperty of the DFT.

This property states that if

$$x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$$

& $$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$$

then $$x_1(n) \, (N) \, x_2(n) = \sum_{n=0}^{N-1} x_1(n) . x_2[(m-n)]_N \xleftrightarrow[N]{DFT} X_1(K).X_2(K)$$

Here $x_1(n)$ (N) $x_2(n)$ means circular convolution of $x_1(n)$ and $x_2(n)$. This property states that multiplication of two DFT is equivalent to circular convolution of their sequence in time domain.

The basic difference between circular convolution and linear convolution is that, in circular convolution, the folding and shifting operations are performed in circular fashion by computing the index of one of the ~~se~~ sequences modulo N. Modulo N operation is not used in linear convolution.

Proof→ By definition of N-points DFTs, ~~are~~ $X_1(K)$ and $X_2(K)$ are given as.

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) \, e^{-j2\pi nK/N} \quad , K=0,1 -- N-1 \quad \text{—} (1)$$

$$X_2(K) = \sum_{\ell=0}^{N-1} x_2(\ell) \, e^{-j2\pi \ell K/N} \quad , K=0,1 -- N-1 \quad \text{—} (2)$$

If we multiply the two DFTs together, the result is a DFT, say $X_3(k)$, of a sequence $x_3(n)$ of length N. Let $X_3(K)$ be equal to multiplication of $X_1(K)$ & $X_2(K)$ i.e.

$$X_3(K) = X_1(K) . X_2(K) \quad ... K=0,1, -- N-1 \quad \text{—} (3)$$

The IDFT of $\{X_3(k)\}$ is

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\cdot X_2(k)\, e^{j2\pi km/N} \quad \text{--- (4)}$$

Let us substitute for $X_1(k)$ & $X_2(k)$ in above equation from equation (1) & (2).

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2(l)\, e^{-j2\pi kl/N}\right] \times e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\cdot\sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] \quad \text{--- (5)}$$

The inner sum in the bracket in (5) has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a=1 \\ \dfrac{1-a^N}{1-a}, & a\neq 1 \end{cases} \quad \text{--- (6)}$$

where a is defined as

$$a = e^{j2\pi(m-n-l)/N}.$$

We observe that $a=1$ when $m-n-l$ is multiple of N. On the other hand, $a^N = 1$ for any value of $a \neq 0$, Hence, eq.(6) reduces to

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m-n+pN = ((m-n))_N,\ p \text{ an integer} \\ 0, & \text{otherwise} \end{cases} \quad \text{--- (7)}$$

If we substitute the result in (7) into (5), we get the desired expression for $x_3(m)$ in the form

$$\boxed{x_3(m) = \sum_{n=0}^{N-1} x_1(n)\cdot x_2((m-n))_N}, \quad m = 0, 1, \dots N-1 \quad \text{--- (8)}$$

The expression in (8) has the form of a convolution sum. Thus the property of circular convolution is proved. Circular convolution of $x_1(n)$ & $x_2(n)$ is denoted by $x_1(n)$ Ⓝ $x_2(n)$ and it is given by equation (8) as

$$x_3(m) = x_1(n) \, Ⓝ \, x_2(n) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2[(m-n)]_N \,, \quad m = 0, 1 \ldots N-1$$

Thus, circular convolution property is proved.

Ex1. Perform the circular convolution of the following sequences:

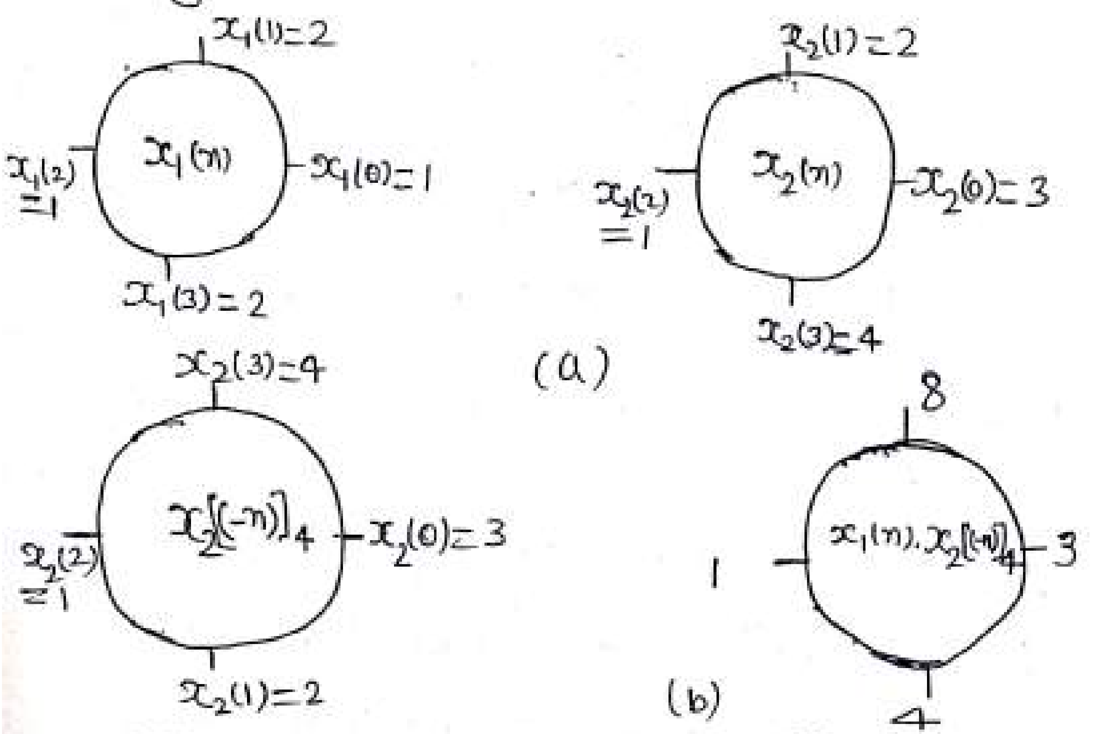
$$x_1(n) = \{\underset{\uparrow}{1}, 2, 1, 2\}$$

$$x_2(n) = \{3, 2, 1, 4\}$$

Soln: Circular convolution is defined as

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2[(m-n)]_N$$

Computation of circular convolution using graph is shown in Fig. 1.



(a)

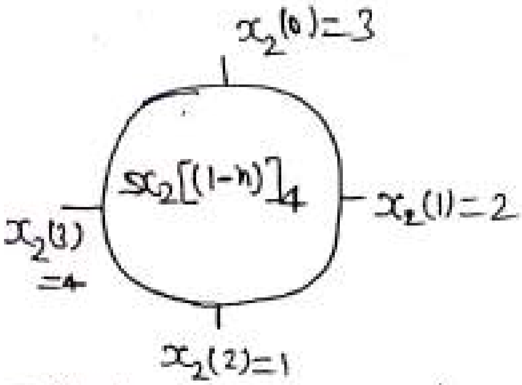
(b)

Folded sequence, $x_2[(-n)]_4$

Product sequence

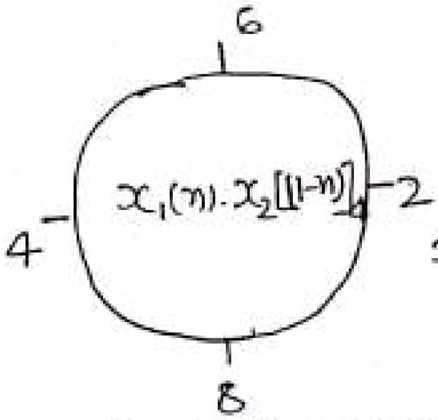
$x_3(0) = 3+8+1+4 = 16$

Fig.1 (b) Circular convolution of two sequences



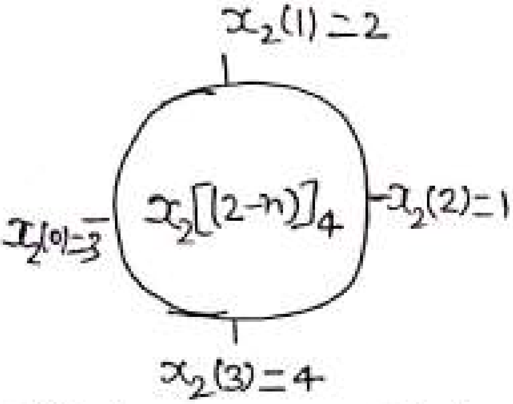
Folded sequence rotated by one units in time.

(c)

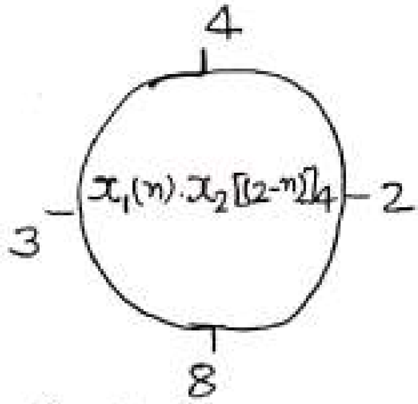


Product sequence

$x_3(1) = 2+6+4+2$
$= 14$

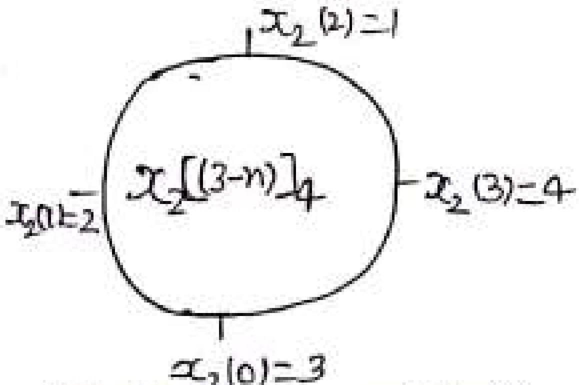


Folded sequence rotated by two units in time

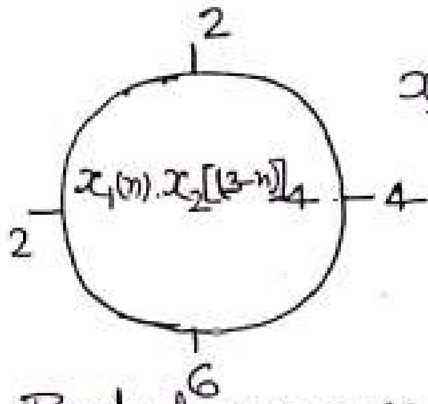


(c) Product sequence.

$x_3(2) = 2+4+3+8$
$= 17$



Folded sequence rotated by three units in time.



$x_3(3) = 4+2+2+6$
$= 14$

(d) Product sequence

Fig 1: Circular convolution of two sequence

From Fig.1, we ~~compu~~ can compute following:

$m=0, \quad x_3(0) = \sum_{n=0}^{3} x_1(n).x_2[(-n)]_4 = 3+8+1+4 = 16$

$m=1, \quad x_3(1) = \sum_{n=0}^{3} x_1(n).x_2[(1-n)]_4 = 2+6+4+2 = 14$

$m=2, \quad x_3(2) = \sum_{n=0}^{3} x_1(n).x_2[(2-n)]_4 = 2+4+3+8 = 17$

$m = 3,\ x_3(3) = \sum_{n=0}^{3} x_1(n).\ x_2[(3-n)]_4 = 4+2+2+6$
$= 14.$

Now circularly convoluted sequence

$$x_3(n) = \{16, 14, 17, 14\}$$

Exp. 2 Compute the $x_3(n)$ corresponding to circular convolution of the sequence $x_1(n)$ and $x_2(n)$ by using DFT and IDFT approach. Given as $x_1(n) = \{2, 1, 2, 1\}$ and $x_2(n) = \{1, 2, 3, 4\}$

Soln → DFT of $x_1(n)$ is given by

$$X_1(K) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi nK/4}, \quad K = 0, 1, 2, 3$$

$$X_1(K) = 2 + e^{-j\pi K/2} + 2e^{-j\pi K} + e^{-j3\pi K/2}$$
$$, K = 0, 1, 2, 3.$$

Thus, $K=0$, $X_1(0) = 2+1+2+1 = 6$

$K=1$, $X_1(1) = 0$

$K=2$, $X_1(2) = 2$

$K=3$, $X_1(3) = 0$

$X_1(K) = \{6, 0, 2, 0\}$

DFT of $x_2(n)$ is given as

$$X_2(K) = \sum_{n=0}^{3} x_2(n) e^{-j2\pi Kn/4}, \quad K = 0, 1, 2, 3$$

$$= 1 + 2e^{-j\pi K/2} + 3e^{-j\pi K} + 4e^{-j3\pi K/2}.$$

Thus, $K=0$, $X_2(0) = 1+2+3+4 = 10$

$K=1$, $X_2(1) = -2+j2$

$K=2$, $X_2(2) = -2$

$K=3$, $X_2(3) = -2-j2$

$X_2(K) = \{10, -2+j2, -2, -2-j2\}$

When we multiply the two DFTs, we obtain the product

$$X_3(k) = X_1(k) \cdot X_2(k)$$
$$= \{6, 0, 2, 0\} \cdot \{10, -2+j2, -2, -2-j2\}$$

$$X_3(k) = \{60, 0, -4, 0\}$$

Now, the IDFT of $X_3(k)$ for computing $x_3(n)$

$$x_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k) e^{j2\pi kn/N}, \quad n = 0, 1, 2, 3$$

$$= \frac{1}{4}\sum_{k=0}^{3} X_3(k) e^{j2\pi kn/4}$$

$$x_3(n) = \frac{1}{4}\left[60 - 4e^{j\pi n}\right] = 15 - e^{j\pi n}$$

$n = 0$, $x_3(0) = 15 - 1 = 14$

$n = 1$, $x_3(1) = 16$

$n = 2$, $x_3(2) = 14$

$n = 3$, $x_3(3) = 16$

$$x_3(n) = \{14, 16, 14, 16\}$$

This is the result of output sequence from circular convolution of two sequences by DFT method.

Use of the DFT in Linear Filtering: Linear filtering operation is implemented with the help of linear convolution. The output $y(n)$ is obtained by convolving impulse response $h(n)$ with input $x(n)$. Now let us see how DFT can be used to implement linear convolution. Hence linear convolution can be computed efficiently with help of DFT.

In this case we seek a frequency domain methodology equivalent to linear convolution. Suppose that we have a finite-duration sequence $x(n)$ of length L which excites an FIR filter system $h(n)$ of length M.

$$x(n) = 0, \quad n < 0 \text{ and } n \geq L$$

$$h(n) = 0, \quad n < 0 \text{ and } n \geq M.$$

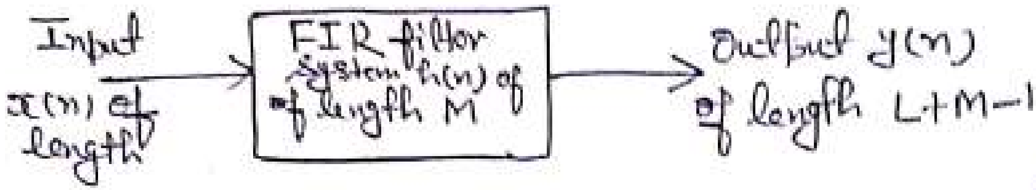


Fig 1. FIR filter which uses linear convolution.

The output sequence $y(n)$ of the FIR filter can be expressed in the time domain as the convolution of $x(n)$ & $h(n)$, that is

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k) \qquad \text{———} \quad (1)$$

Since $h(n)$ & $x(n)$ are finite duration sequences, their convolution is also finite in duration.

The frequency-domain equivalent to (1) is

$$Y(\omega) = X(\omega)\, H(\omega). \qquad \text{———} \quad (2)$$

Here $H(\omega)$ is Fourier transform of $h(n)$ and $X(\omega)$ is Fourier transform of $x(n)$.

If the sequence $y(n)$ is to be represented uniquely in the frequency domain by samples of its spectrum $Y(\omega)$ at a set of discrete frequencies, the number of distinct samples must equal or exceed $L+M-1$. therefore, a DFT of size $N \geq L+M-1$ is required to represent $\{y(n)\}$ in the frequency domain.

Now if

$$Y(K) \equiv Y(\omega)\Big|_{\omega = 2\pi K/N}, \quad K = 0, 1, \ldots N-1$$

$$= X(\omega)\, H(\omega)\Big|_{\omega = 2\pi K/N}, \quad K = 0, 1 \ldots N-1$$

then

$$Y(K) = X(K)\, H(K), \quad K = 0, 1, \ldots N-1 \quad \text{---} (3)$$

This shows that multiplying the N-point DFTs of $x(n)$ and $h(n)$, we get DFT $Y(K)$. This DFT represent $y(n)$ uniquely if $N \geq L+M-1$. Hence $y(n)$ can be obtained by taking IDFT of $Y(K)$ i.e

$$y(n) = \text{IDFT}\{Y(K)\} \quad \text{---} (4)$$

$$= \text{IDFT}\{X(K).H(K)\}, \quad K = 0, 1 \ldots N-1$$

Here observe that $y(n)$ can be obtained with help of DFT. Thus linear convolution is implemented by DFT.