from $dz = p\,dx + q\,dy$

$\therefore\ dz = p\,dx + ap\,dy$

$dz = p\,(dx + a\,dy)$

Let $z = f(x+ay)$ as a trial solⁿ of (1)

where a is an arbitrary constant.

$\therefore\ z = f(u)$, $u = x+ay$

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{dz}{du}$

$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} = a\frac{dz}{du}$

$\therefore$ eqⁿ (1) reduces to $f\left(z, \frac{\partial z}{\partial u}, a\frac{\partial z}{\partial u}\right) = 0$

which is an ordinary differential eqⁿ of order one. Integrating it, we may get the complete integral.

Q. find a complete integral of

$9(p^2 z + q^2) = 4.$

Solⁿ The given eqⁿ is of the form $f(p,q,z)=0$

$\therefore$ Let $p = \frac{\partial z}{\partial u}$, $q = a\frac{\partial z}{\partial u}$

where $u = x+ay$

$\therefore$ The given eqⁿ becomes.

$$9\left[z\left(\frac{\partial z}{\partial u}\right)^2 + a^2\left(\frac{\partial z}{\partial u}\right)^2\right] = 4$$

or $$9(z+a^2)\left(\frac{\partial z}{\partial u}\right)^2 = 4$$

or $$\left(\frac{\partial z}{\partial u}\right)^2 = \frac{4}{9(z+a^2)}$$

$$\frac{\partial z}{\partial u} = \frac{2}{3\sqrt{z+a^2}}$$

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or $\cancel{\frac{3}{2}du} =$

$$\frac{3}{2}\sqrt{z+a^2}\,dz = du$$

integrating, we get

$$\frac{3}{2}\frac{(z+a^2)^{3/2}}{3/2} = u+b$$

or $(z+a^2)^{3/2} = x+ay+b$

or $(z+a^2)^3 = (x+ay+b)^2$

Which is the complete integral of the given eqn.

<u>Problem</u> find a complete integral of

1) $z^2(p^2z^2+q^2)=1$.

2) $pz = 1+q^2$.

<u>Case (ii)</u> <u>Equations of the form $f(x,p) = g(y,q)$</u>

Let $f(x,p) = g(y,q) = a$, a ~~const~~ constant.

$\therefore\ p = f_1(x,a)\ ,\ q = f_2(y,a)$

$\because\ dz = p\,dx + q\,dy$

$= f_1(x,a)dx + f_2(y,a)\,dy$

integrating $z = \int f_1(x,a)\,dx + \int f_2(y,a)\,dy + b$

Which is complete integral

we can obtain its general integral.

There is no singular integral.

Q find a complete integral of $p^2+q^2 = x+y$

Sol^n $\because$ $p^2+q^2 = x+y$

or $p^2 - x = y - q^2 = a$

$\Rightarrow$ $p = \sqrt{x+a}$, $q = \sqrt{y-a}$

$\because$ $dz = p\,dx + q\,dy$

$= \sqrt{x+a}\,dx + \sqrt{y-a}\,dy$

integrating,

$$z = \frac{(x+a)^{3/2}}{3/2} + \frac{(y-a)^{3/2}}{3/2} + b$$

or $$\boxed{z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b}$$

Q find a complete integral of $z(p^2-q^2) = x-y$

Sol^n We can write the given eq^n as

$$\left[\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2\right] = x-y \quad \text{— (1)}$$

Putting $\sqrt{z}\,dz = du$, on integration

$\Rightarrow$ $u = \frac{z^{3/2}}{3/2}$ or $\boxed{u = \frac{2}{3}z^{3/2}}$

$\therefore$ $\sqrt{z}\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} = P$ (say)

$\sqrt{z}\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} = Q$ (say)

$\therefore$ eq^n (1) reduces to $P^2 - Q^2 = x-y$

or $P^2 - x = Q^2 - y = a$

$\Rightarrow$ $P = \sqrt{a+x}$, $Q = \sqrt{a+y}$

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$\therefore$

$$du = P\,dx + Q\,dy \qquad (\text{as } dz = p\,dx + q\,dy)$$

$$= \sqrt{a+x}\,dx + \sqrt{a+y}\,dy$$

integrating,

$$u = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(a+y)^{3/2} + b_1$$

or $$\frac{2}{3} z^{3/2} = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(a+y)^{3/2} + b_1$$

or $$\boxed{z^{3/2} = (a+x)^{3/2} + (a+y)^{3/2} + b}$$

Case IV Equation of the form

$$z = px + qy + f(p,q) \quad \text{———} (1)$$

which is analogous to Clairaut's form.

Since general solⁿ of Clairaut's eqⁿ ~~$y = p(x)$~~ $y = px + f(p)$; $p = \frac{dy}{dx}$

is $$y = cx + f(c)$$

Similarly the complete integral of Clairaut's eqⁿ (1) is

$$z = ax + by + f(a,b).$$

~~which is obtained by~~

Since from (1)

$$F(x,y,z,p,q) = px + qy - z + f(p,q)$$

$$\frac{\partial F}{\partial x} = p, \quad \frac{\partial F}{\partial y} = q, \quad \frac{\partial F}{\partial z} = -1, \quad \frac{\partial F}{\partial p} = x + \frac{\partial f}{\partial p}$$

$$\frac{\partial F}{\partial q} = y + \frac{\partial f}{\partial q}.$$

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from Charpit's eq^n $dp = 0 \Rightarrow p = a$

& $dq = 0 \Rightarrow q = b$

$\therefore$ $\boxed{z = ax + by + f(a,b)}$

We can obtain its general integral as in previous cases.

Singular Integral: Since the complete integral be

$$F = z - ax - by - f(a,b) = 0 \quad \text{——(1)}$$

$$\frac{\partial F}{\partial a} = 0 \Rightarrow x + \frac{\partial f}{\partial a} = 0 \quad \text{——(2)}$$

$$\frac{\partial F}{\partial b} = 0 \Rightarrow y + \frac{\partial f}{\partial b} = 0 \quad \text{——(3)}$$

Singular integral is obtained by eliminating a, b from (1), (2) & (3).

Q find a complete integral of

$$z = px + qy + p^2 + q^2$$

Sol^n Since it is of clairaut's form, so its complete integral is

$$z = ax + by + a^2 + b^2,$$

where a and b are arbitrary constants.

Q find the singular integral of

$$z = px + qy + \log pq$$

Sol^n The complete integral of the ~~eq~~ given eq^n be $z = ax + by + \log(ab)$, ——(1)

where a and b are arbitrary constants.

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Differentiate (1) partially w.r.t 'a' and 'b', we get

$$0 = x + \frac{1}{a} \Rightarrow a = -1/x \quad —(2)$$

and $$0 = y + \frac{1}{b} \Rightarrow b = -1/y \quad —(3)$$

Eliminating a, b between (1), (2) & (3), we get.

$$z = \left(-\frac{1}{x}\right)x + \left(-\frac{1}{y}\right)y + \log\left(-\frac{1}{x}\right)\left(-\frac{1}{y}\right)$$

$$= -2 + \log\frac{1}{xy}$$

or $$\boxed{z = -2 - \log xy}$$

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