Differential eqⁿ 2 (1) (Sem 4)

The Integrals of the Non-linear equations

Any relation, which contains as many arbitrary constants as there are independent variables and is a solution of a partial differential equation of the first order is called a <u>complete solution</u> or <u>complete integral</u> of that equation.

If $F(x, y, z, p, q) = 0$ be partial differential equation with z as dependent variable and x and y are independent variables,

$$p = \frac{\partial z}{\partial x} \quad , \quad q = \frac{\partial z}{\partial y}$$

Then <u>complete integral</u> of this differential eqⁿ is

$$f(x, y, z, a, b) = 0 \quad \text{———} \quad (1)$$

where a and b are arbitrary constant

<u>Particular integral</u> is obtained by giving particular values to a and b in (1).

<u>Singular Integral</u>: The equation of the envelope of the surfaces (1) can be obtained by eliminating a and b between the eqⁿs

$$f = 0 , \quad \frac{\partial f}{\partial a} = 0 \quad , \quad \frac{\partial f}{\partial b} = 0$$

This eqⁿ of envelope is called the singular integral of the differential eqⁿ.

<u>General Integral</u>: If (1), one constant is a function of the other i.e $b = \phi(a)$, then this equation becomes

$$f(x, y, z, a, \phi(a)) = 0$$

The equation of the envelope of family of this surface is also solⁿ of p.d.e.

which is general integral of p.d.e. corresponding to (1).

Solution of Non-linear P.D.E of order One

Charpit's Method

Let the given partial differential equation be

$$f(x, y, z, p, q) = 0 \quad \text{——— (1)}$$

Since z depends on x and y. i.e $z = z(x, y)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

or $$dz = p\,dx + q\,dy \quad \text{——— (2)}$$

Charpit's auxiliary equation

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}}$$

$$= \frac{dy}{-\frac{\partial f}{\partial q}}.$$

Q find a complete integral of the equation

$$2zx - px^2 - 2qxy + pq = 0$$

Sol^n The given differential eq^n is

$$f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0 \quad \text{—(1)}$$

$\therefore \quad \frac{\partial f}{\partial x} = 2z - 2px - 2qy$ | $\frac{\partial f}{\partial q} = -2xy + p$

$\frac{\partial f}{\partial y} = -2qx$ | $\frac{\partial f}{\partial z} = 2x$

$\frac{\partial f}{\partial p} = -x^2 + q$

So Charpit's auxiliary equations are

$$\frac{dp}{2z-2px-2qy+p.2x} = \frac{dq}{-2qx+q.(2x)}$$

$$= \frac{dz}{-p(-x^2+q)-q(-2xy+p)} = \frac{dx}{-(-x^2+q)} = \frac{dy}{-(-2xy+p)}$$

or $$\frac{dp}{2z-2qy} = \frac{dq}{0} = \frac{dz}{px^2-pq+2qxy-pq}$$

$$= \frac{dx}{x^2-q} = \frac{dy}{2xy-p} \quad \text{———} (2)$$

$\therefore$ $dq = 0 \Rightarrow \boxed{q = a}$ (a constant) ——(3)

from (1) and (3), we get

$\cancel{2zx - px^2 - 2qxy +}$

$$2zx - px^2 - 2axy + ap = 0$$

or $$p = \frac{2x(z-ay)}{x^2-a} \quad \text{———} (4)$$

Putting these values of p and q in

$$dz = p\,dx + q\,dy$$

$$= \frac{2x(z-ay)}{x^2-a}dx + a\,dy$$

or $$\frac{dz-a\,dy}{z-ay} = \frac{2x}{x^2-a}dx$$

integrating. $\log(z-ay) = \log(x^2-a) + \log b$

where b is a constant.

or $\boxed{z = ay + b(x^2-a)}$ which is a complete integral.

Special Methods

1) **Equations involving only p and q and no x, y, z.**

Let the equation be $f(p, q) = 0$ ——— (1)

$\therefore \frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial z} = 0$.

$\therefore$ Charpit's eq auxiliary eq^{n}s are

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial z}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$\therefore$ $dp = 0 \Rightarrow \boxed{p = a}$.

$dq = 0 \Rightarrow \boxed{q = b}$

$\therefore$ from $dz = p\,dx + q\,dy$

$dz = a\,dx + b\,dy$

or $\boxed{z = ax + by + c}$ ——— (2)

where a and b are connected by

$f(a, b) = 0$. {from (1)}

$\Rightarrow b = \phi(a)$

$\therefore$ complete integral of (1) is

$$\boxed{z = ax + \phi(a)y + c}$$

Putting $c = \psi(a)$, where ψ is arbitrary fn

$\therefore$ $z = ax + \phi(a)y + \psi(a)$ ——(3)

Differentiate it partially w.r.t. a, we get

$0 = x + \phi'(a)y + \psi'(a)$ —— (4)

General integral is obtained by eliminating a in (3) & (4).

Singular Integral Singular integral is obtained by eliminating a and c between
$z = ax + \phi(a)y + c$ and its partial derivatives w.r.t a and c

i.e $0 = x + \phi'(a)y$

and $0 = 1$, which is inconsistent.

$\Rightarrow$ In this case, there is no singular integral.

Q find the complete integral of $q = 3p^2$.

Soln given eqn is of the form $f(p,q) = 0$

$\therefore$ $p = a$ and $q = b$ (As done earlier)

$\therefore$ $dz = p\,dx + q\,dy$

or $dz = a\,dx + b\,dy$

or $z = ax + by + c$

$\because$ $q = 3p^2$, $\because p = a$, $q = b$

$\Rightarrow$ $b = 3a^2$

$\therefore$ $z = ax + 3a^2y + c$

where a and c are arbitrary constants.

Q Solve $x^2p^2 + y^2q^2 = z^2$

Soln The given eqn can be written as

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \qquad (1)$$

Since $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Put $\frac{dx}{x} = du$, $\frac{dy}{y} = dv$, $\frac{dz}{z} = dw$

$\Rightarrow$ $u = \log x$, $v = \log y$, $w = \log z$

$$\therefore\ p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\cdot 0$$

$$= \frac{1}{x}\frac{\partial z}{\partial u} \Rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= \frac{1}{z}\frac{\partial z}{\partial u} \quad \text{--- (2)}$$

Similarly $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$

$$= \frac{1}{y}\frac{\partial z}{\partial v}$$

$$\Rightarrow \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v} \quad \text{--- (3)}$$

$$= \frac{1}{z}\frac{\partial z}{\partial v}$$

$$\Rightarrow \quad \frac{\partial w}{\partial u} = \frac{x}{z}\frac{\partial z}{\partial x} \quad \& \quad \frac{\partial w}{\partial v} = \frac{y}{z}\frac{\partial z}{\partial y}$$

Substituting these in (1), we get,

$$\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = 1 \quad \text{or} \quad \cancel{p}\ P^2 + Q^2 = 1$$

which is of the form $f(P, Q) = 0$

where $P = \frac{\partial w}{\partial u}$

$Q = \frac{\partial w}{\partial v}$

$\therefore$ complete integral be

$$w = au + bv + c_1 \quad \text{——— (4)}$$

where $a^2 + b^2 = 1$ $\left\{ \begin{array}{l} \because P = a,\ Q = b \\ \ \&\ P^2 + Q^2 = 1 \end{array} \right.$

~~on log~~ Substituting u, v, w in (4), we get

$$\log z = a\log x + \sqrt{1-a^2}\,\log y + c_1$$

or $\log z = a\log x + \sqrt{1-a^2}\,\log y + \log c$

or $\boxed{z = c\, x^a\, y^{\sqrt{1-a^2}}}$ ——— (5)

General integral: Let $c = \phi(a)$

$$z = \phi(a)\cdot x^a\, y^{\sqrt{1-a^2}} \qquad \text{——— (6)}$$

general integral is obtained by eliminating a between (6) and its partial derivative w.r.t 'a'.

Singular Integral: Singular integral is obtained by eliminating a and c between (5) and its partial derivatives w.r.t. 'a' and 'c'.

(You can obtain general and singular integrals of the given differential eqn after a little computation, as described.)

Case II Equations involving only p, q and z

i.e $f(z, p, q) = 0$ ——— (1)

$\therefore \dfrac{\partial f}{\partial x} = 0,\ \dfrac{\partial f}{\partial y} = 0,\ \cancel{\dfrac{\partial f}{p}}$

Charpit's eqns are

$$\frac{dp}{p\frac{\partial f}{\partial z}} = \frac{dq}{q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

Taking first two members, we get

$$\frac{dp}{p} = \frac{dq}{q}$$

integrating, we get $p = aq$, where a is a constant.
or $q = qp$

[Contd. to D.E - 3