Differential Equation-1 (B.Sc Sem 4)

A partial differential equation, which is linear in p and q is of type

$$Pp + Qq = R, \quad \text{———} (1)$$

where P, Q, R are functions of x, y, z.

The equation (1) is referred to as Lagrange's equation.

Thm The general solⁿ of the linear partial differential eqⁿ $Pp + Qq = R$ is $f(u, v) = 0$ where f is an arbitrary funⁿ and $u(x, y, z) = c$ and $v(x, y, z) = c_2$ form a solⁿ of the equations.

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}} \quad \text{———} (2)$$

Eqⁿ (2) is called Lagrange's auxiliary eqⁿs.

The equation $Pp + Qq = R$ represents a family of surfaces Orthogonal to the family of surfaces represented by $P\,dx + Q\,dy + R\,dz = 0$

Since the direction ratio's of the normal at the point (x, y, z) to a surface of the family $Pp + Qq = R$ are $p, q, -1$

Also direction ratio's of the normal at (x, y, z) to the surface of family $P\,dx + Q\,dy + R\,dz = 0$ are P, Q, R.

Since $Pp + Qq + (-1)R = 0$

$\Rightarrow$ Two lines whose direction ratio's are $p, q, -1$ and P, Q, R are perpendicular.

Hence the surfaces represented by $Pp + Qq = R$ are orthogonal to the surface represented by $P\,dx + Q\,dy + R\,dz = 0$

Q Solve $(y+z)p + (z+x)q = x+y$

Sol^n The Lagrange's auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \qquad \because P = y+z,\ Q = z+x,\ R = x+y$$

$$\therefore \frac{dx-dy}{(y+z)-(z+x)} = \frac{dy-dz}{(z+x)-(x+y)} = \frac{dx+dy+dz}{y+z+z+x+x+y}$$

$$\text{or} \quad \frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)} = \frac{dx+dy+dz}{2(x+y+z)}$$

Taking first two members, we get

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)}$$

integrating $\quad -\log(x-y) = -\log(y-z) - \log c_1$

or $\quad x-y = c_1(y-z)$

i.e $\boxed{u = \frac{x-y}{y-z} = c_1}$

Taking last two members, we get

$$\frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)}$$

or $\quad -2\log(y-z) = \log(x+y+z) - \log c_2$

or $\boxed{\cancel{v = \frac{x+y+z}{(y-z)^2} = c_2}}$ $\boxed{v = (x+y+z)(y-z)^2 = c_2}$

$\therefore$ General sol^n of given eq^n

$$f\left[\left(\frac{x-y}{y-z}\right), (x+y+z)(y-z)^2\right] = 0.$$

Q. Find the equation of the integral surface of the differential equation

$$2y(z-3)p + (2x-z)q = y(2x-3) \quad \text{——— (1)}$$

which passes through the circle $z=0$, $x^2+y^2=2x$.

Sol^n From given eq^n

$$P = 2y(z-3),\ Q = 2x-z,\ R = y(2x-3)$$

$\therefore$ Lagrange's auxiliary eq^{n}s are

$$\frac{dx}{2y(z-3)} = \frac{dy}{(2x-z)} = \frac{dz}{y(2x-3)}$$

Taking first and third members, we get

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)}$$

or $(2x-3)\,dx = 2(z-3)\,dz$

integrating $x^2 - 3x = z^2 - 6z + c_1$

or $\boxed{x^2 - z^2 - 3x + 6z = c_1}$ ——— (2)

Using $\frac{1}{2}, y, -1$ as multipliers, we get

$$\frac{\frac{1}{2}dx + y\,dy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = \frac{\frac{1}{2}dx + y\,dy - dz}{yz - 3y + 2xy - zy - 2xy + 3y}$$

$\Rightarrow \frac{1}{2}dx + y\,dy - dz = 0$

Integrating, $\frac{x}{2} + \frac{y^2}{2} - z = \frac{c_2}{2}$

or $\boxed{x + y^2 - 2z = c_2}$ ——— (3)

Since parametric eq^n of the given circle is
$$x = t, \quad y = \sqrt{2t - t^2}, \quad z = 0$$
Putting these values in (2) and (3), we get
$$t^2 - 3t = c_1 \quad \text{and} \quad t + (2t - t^2) = c_2$$
eliminating 't' between these eq^{n}s (by adding these two), we get
$$c_1 + c_2 = 0$$
$\therefore$ The required integral surface is

$\cancel{(x+y^2-2z)+(}$ (from (2) & (3))
$$(x^2 - z^2 - 3x + 6z) + (x + y^2 - 2z) = 0$$
or $$\boxed{x^2 + y^2 - z^2 - 3x + 4z = 0}$$

Q Find the family orthogonal to
$$f\left(z(x+y)^2, x^2 - y^2\right) = 0 \quad \text{——— (1)}$$
Sol^n Let $u = z(x+y)^2$, $v = x^2 - y^2$ —— (2)

Differentiate partially (1) w.r.t x ~~and y,~~

~~∂f~~ Since eq^n (1) be $f(u, v) = 0$
$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0$$
or $$\frac{\partial f}{\partial u}\left[2(x+y)z + (x+y)^2\frac{\partial z}{\partial x}\right] + \frac{\partial f}{\partial v}\left[2x - 0\right] = 0$$
$$\frac{\partial f}{\partial u}\left[2(x+y)z + (x+y)^2 p\right] + 2x\frac{\partial f}{\partial v} = 0$$
or $$\frac{\partial f}{\partial u} \Big/ \frac{\partial f}{\partial v} = \frac{-2x}{2(x+y)z + (x+y)^2 p} \quad \text{——— (3)}$$

Similary differentiate (1) partially w.r.t y,

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial f}{\partial u}\left[2(x+y)z + (x+y)^2 q\right] + \frac{\partial f}{\partial v}(-2y) = 0$$

$$\therefore \quad \frac{\partial f}{\partial u}\Big/\frac{\partial f}{\partial v} = \frac{2y}{2(x+y)z + (x+y)^2 q} \qquad \text{---(4)}$$

from (3) and (4), we get

$$\frac{-2x}{2z(x+y) + (x+y)^2 p} = \frac{2y}{2(x+y)z + (x+y)^2 q}$$

or $\quad 2y\left[2z(x+y) + (x+y)^2 p\right] = -2x\left[2(x+y)z + (x+y)^2 q\right]$

or $\quad y(x+y)^2 p + x(x+y)^2 q = -2z(x+y)^2$

or $\quad yp + xq = -2z$

which is of the form $Pp + Qq = R$

where $\quad P = y,\ Q = x,\ R = -2z$

Hence the differential equation of the family of surfaces orthogonal to the given family is

$$P\,dx + Q\,dy + R\,dz = 0$$

or $\ y\,dx + x\,dy - 2z\,dz = 0 \quad$ or $\quad d(xy) - 2z\,dz = 0$

Integrating, we get

$$\boxed{xy - z^2 = c}$$

which is required family.

Problems

1. Find the integral surface of the linear partial differential eq^n

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains the line $x+y=0$, $z=1$.

2. Find the family of surfaces orthogonal to the family of surfaces given by the differential equation

$$(y+z)p + (z+x)q = x+y.$$