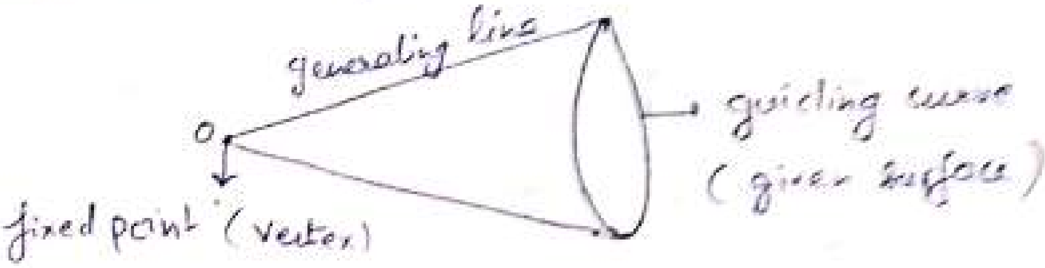
Cone UG Sem II Unit III

Def A cone is a surface generated by a variable line which passes through a fixed point and intersects a given curve or touching a given surface. The



To find the eq of a cone.

Let (α, β, γ) be a point on the cone. The equation of its generator is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ — (1)

Let base or guiding curve of the conic is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \ z = 0$ — (2)

The line (1) meets the plane $z=0$ i.e. put $z=0$ in (1) and get a point $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$ which must lie on (2)

$\therefore a\left(\alpha - \frac{l\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{l\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + b\left(\beta - \frac{m\gamma}{n}\right) +$

$2g\left(\alpha - \frac{l\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0$

eliminate l, m, n by using (1) $\therefore$ the cone is generated by (1) when it always intersect or touches the conic i.e.

$a\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right)^2 + 2h\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right)\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right)$

$+ b\left(\beta - \frac{y-\beta}{z-\gamma}\right)^2 + 2g\left(\alpha - \frac{x-\alpha}{z-\gamma}\cdot\gamma\right) + 2f\left(\beta - \frac{y-\beta}{z-\gamma}\cdot\gamma\right) + c = 0$

or $a(\alpha z-\gamma x)^2 + 2(\alpha z-\gamma x)(\beta z-\gamma y)h + b(\beta z-\gamma y)^2$

$+ 2g\frac{(\alpha z-\gamma x)}{(z-\gamma)} + 2f\frac{(\beta z-\gamma y)}{(z-\gamma)} + c(z-\gamma)^2 = 0$

equation of the cone when vertex is origin

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

This is a homogeneous equation in second degree

The homogeneous equation of the cone is satisfied by the direction cosines of the generators

ie $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is the generator at origin

$\therefore$ $x = l\lambda$, $y = m\lambda$, $z = n\lambda$

$\therefore$ $f(l,m,n) = al^2 + bm^2 + cn^2 + 2fmn + 2gln + 2hlm = 0$

ex Find the equation of the cone whose vertex is at the point $(-1, 1, 2)$ and whose guiding curve is

$3x^2 - y^2 = 1$; $z = 0$.

Sol.

Let eq of the generator of the cone is

$$\frac{x+1}{l} = \frac{y-1}{m} = \frac{z-2}{n}$$

it passes through the plane $z=0$ and we get a point $\left(-1-\frac{2l}{n},\ 1-\frac{2m}{n},\ 0\right)$ which must lie on $3x^2 - y^2 = 1$ ie

$$3\left(-1-\frac{2l}{n}\right)^2 - \left(1-2\frac{m}{n}\right)^2 = 1$$

eliminating $\frac{l}{n}$ and $\frac{m}{n}$ by ~~the~~ $\frac{x+1}{z-2}$, $\frac{y-1}{z-2}$

$$3\left(1+2\frac{x+1}{z-2}\right)^2 - \left(1-2\frac{y-1}{z-2}\right)^2 = 1$$

or $3(z+2x)^2 - (z-2y)^2 = 1(z-2)^2$

$12x^2 - 4y^2 + z^2 - 4yz + 12zx + 4z - 4 = 0$

Ex Prove that the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, where $l^2 - 2m^2 + 3n^2 = 0$ is a generator of the cone. $x^2 - 2y^2 + 3z^2 = 0$

Sol. If $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be the generator then eq of the cone having vertex with origin

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

ie $a = 1$, $b = -2$, $c = 3$, $f = 0 = g = h = 0$

$\therefore f(l, m, n) = l^2 - 2m^2 + 3n^2$.

To find the condition for the general equation of second degree to represent a cone and to find its vertex

Let the general equation of second degree in x, y, z

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$

Condition for cone

$$\Delta' = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

To find vertex :— Write the equation in the form

$f(x,y,z\,t) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy$
$+2uxt+2vyt+2wzt+dt^2=0$

Now diff it w.r.t. x, y, z and t we get four equations with four variables. Solve these equations to find the vertex of the cone to put $t=1$

Remark. If $\Delta'=0$ and $\Delta=0$ are satisfied simultaneously the equation of the cone represents a pair of planes.

ex Prove that the equation

$4x^2-y^2+2z^2-3yz+2xy+12x-11y+6z+4=0$

represents a cone whose vertex is $(-1,-2,-3)$.

Sol. $f(x,y,z,t)=4x^2-y^2+2z^2-3yz+2xy+12xt-11yt+6zt+4t^2=0$

$\frac{\partial f}{\partial x} = 8x+2y+12t = 0$

$\frac{\partial f}{\partial y} = -2y-3z+2x-11t = 0$

$\frac{\partial f}{\partial z} = 4z-3y+6t = 0$

$\frac{\partial f}{\partial t} = 12x-11y+6z+8t=0$

On solving as put $t=1$ we get $(-1,-2,-3)$

Now we can see the above four equation are consistent.

Therefore this equation represents a cone. i.e Δ' must be zero.

Ex Find the equation of the cone with vertex at origin and pass through the curves of intersection

1. $ax^2 + by^2 = 2z$, $lx + my + nz = p$.

Sol

The equation of the plane can be written as

$$\frac{lx + my + nz}{p} = 1$$

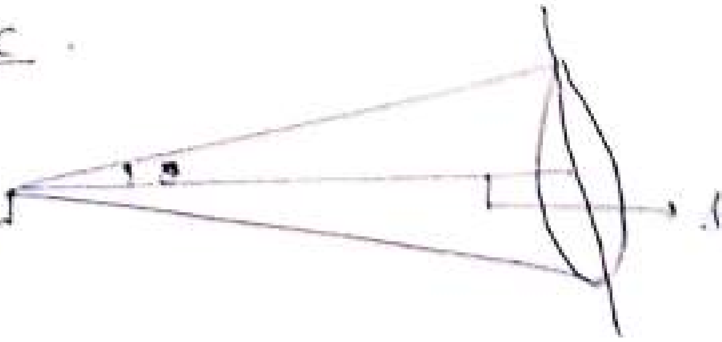
Now to make $ax^2 + by^2 = 2z\left(\frac{lx + my + nz}{p}\right)$ homogeneous

ie $apx^2 + pby^2 - 2lzx - 2myz - 2nz^2 = 0$

which is a homogeneous eq. ie equation of a cone, vertex origin

Right Circular Cone RCC

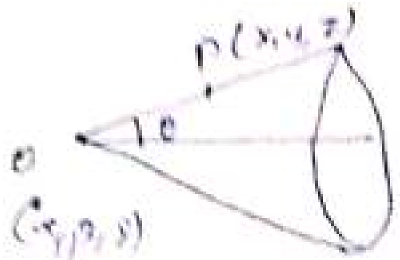
Def. A surface generated by a st. line which passes through a fixed point and makes a constant angle with a fixed line through that point is called RCC.



Remarks → The section of a RCC by a plane ⊥ to its axis is a circle.

A cone with base curve as a circle is not necessarily a RCC.

To find the equation of RCC



Let eq of the axis is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

Let P be any point on the cone the direction cosines of the OP are prop to $x-\alpha$, $y-\beta$, $z-\gamma$.

Hence $$\cos\theta = \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{l^2+m^2+n^2}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

Ex - Find the eq of cone when vertex is origin, $\theta = 60°$ and eq of the axis is $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

Sol. $$\cos\frac{60}{2} = \frac{1(x-0) + 2(y-0) + 3(z-0)}{\sqrt{3}\sqrt{x^2+y^2+z^2}}$$

$$\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{(x+2y+3z)^2}{(\sqrt{3}\sqrt{x^2+y^2+z^2})^2}$$

$$9(x^2+y^2+z^2) = 4(x^2+4y^2+9z^2+4xy+12yz+6zx)$$

Ex Find the equation of a RCC which passes through the three lines drawn from the origin with direction ratios 1, 2, 2 ; 2, 1, -2 , 2, -2, 1

steps →

(i) First write down the equation of lines.

(ii) Find the angle between them

(iii) Find the equation of axis

(iv) Using the formula for RCC.

1. Tangent line - A line which cuts the cone in two coincident points is called the tangent line.

Equation of tangent plane of a Cone.

When a line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ intersect a cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

at two points. If points coincide then locus of the tangent lines to a cone is a point it is called the tangent plane at the point

ie $x(a\alpha + h\beta + g\gamma) + y(h\alpha + b\beta + f\gamma)$
$+ z(g\alpha + f\beta + c\gamma) = 0.$

Remarks - 1. A tangent plane at any point of a cone passes through its vertex (ie contant term is 0)

(ii) The vertex of the cone through which all the tangent planes pass is called the singular point of the surface

(iii) The tangent plane touches the cone at every point of the generator.

Condition of the tangency of the plane $ux + vy + wz = 0$ — (1)

Proof. Equation of a tangent plane at the point (α, β, γ) of the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

is $x(a\alpha + h\beta + g\gamma) + y(h\alpha + b\beta + f\gamma) + z(g\alpha + f\beta + c\gamma) = 0$ — (2)

Comparing the coeff from (1) & (2) we get

Let P be any point on the cone the direction cosines

$$\frac{u}{a\alpha+h\beta+g\gamma} = \frac{v}{h\alpha+b\beta+f\gamma} = \frac{w}{g\alpha+f\beta+c\gamma} = \frac{1}{\lambda}$$

or
$$a\alpha+h\beta+g\gamma - u\lambda = 0$$
$$h\alpha+b\beta+f\gamma - v\lambda = 0$$
$$g\alpha+f\beta+c\gamma - w\lambda = 0$$

Also $u\alpha+v\beta+w\gamma = 0$

eliminating $\alpha, \beta, \gamma, \lambda$ among these equations we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0$$

is the required condition of tangency.

Now if we expand the above determinant we have

$$Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Guw + 2Huv = 0$$

where $A = bc - f^2$ $\quad B = ac - g^2$ $\quad C = ab - h^2$

$F = gh - af$ $\quad G = hf - bg$ $\quad H = gf - ch$

Reciprocal Cone –

Two cones which are such that each is the locus of the normals drawn through the vertex to the tangent planes of the other, is called reciprocal cone.

To find the equation of reciprocal cone of a given cone.

Let $f(x, y, z) = 0$ be the eq of a cone passing through origin. Let $ux + vy + wz = 0$ be the plane.

Then we have a eq.

$$Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Guw + 2Huv = 0 \quad \text{—(1)}$$

1. Now the equation of normal to the plane passing through the vertex $(0,0,0)$ of the given cone

$$f(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

is $\frac{x}{u} = \frac{y}{v} = \frac{z}{w} = \lambda$ ——— (2)

Eliminating u, v, w by (1) & (2) we get

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad \text{——— (3)}$$

is the required eq of the reciprocal cone.
A, B, C, F, G and H are defined as above.

Now to find the equation of reciprocal cone of (3).

Any tangent plane to (3) is $ux + vy + wz = 0$

where

$$A'u^2 + B'v^2 + C'w^2 + 2F'vw + 2G'uw + 2H'uv = 0$$

where $A' = BC - F^2 = a\Delta \quad B' = CA - G^2 = b\Delta$

$C' = AB - H^2 = c\Delta \,, \quad F' = GH - AF = f\Delta$

$G' = HF - BG = g\Delta \,, \quad H' = GF - CH = h\Delta$

Hence reciprocal cone of (3) is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

where $\Delta \neq 0$, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Ex Find the equation of lines in which the plane $3x+4y+z=0$ cuts the cone $15x^2-32y^2-7z^2=0$

Sol. Since eq of one of the line is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

$\because$ line lie on the plane $\therefore$

$3l+4m+n=0$ and $15l^2-32m^2-7n^2=0$

$15l^2-32m^2-7(3l+4m)^2=0$

Find the values of ~~relation between~~ l/m & l/n

i.e $\frac{l}{m}=\frac{-3}{2}, \frac{2}{-1}$, $\frac{l}{n}=\frac{-3}{1}, \frac{2}{-2}$

$\therefore$ The equation of lines are

$$\frac{x}{-3}=\frac{y}{2}=\frac{z}{1} \quad . \quad \frac{x}{2}=\frac{y}{-1}=\frac{z}{-2}$$

Ex Prove that the perpendiculars drawn from the origin to the tangent planes to the cone

$$3x^2+4y^2+5z^2+2yz+4zx+6xy=0 \quad \text{—(1)}$$

lie on the cone

$$19x^2+11y^2+3z^2+6yz-10zx-26xy=0$$

Sol According to def of reciprocal cone we want to find the reciprocal cone of (1).

Equation of Reciprocal cone of (1) is

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \quad \text{—(2)}$$

$A=bc-f^2=20-1=19$

$B=ac-g^2=15-4=11$

$C=ab-h^2=12-9=3$

$F=gh-af=6-3\times1=3$

$G=hf-bg=3-8=-5$

$H=fg-ch=2-15=-13$

Substituting the values in (2) and get the result.

1. To find the condition of a cone with three mutually perpendicular generators

Let $f(x,y,z)=0$ be the cone with vertex (origin)

Let $ux+vy+wz=0$ cut the cone in two $\perp$ generators.

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the lines of intersection of the plane and the cone. Then

$$ul+vm+wn=0 \quad \text{and} \quad f(l,m,n)=0$$

eliminating n between these two relations and get.

$$\frac{l_1l_2}{cv^2+bw^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2gwu}=\frac{n_1n_2}{bu^2+av^2-2huv}$$

The lines of intersection of the plane and the cone are perpendicular if

$$u^2(b+c)+v^2(c+a)+w^2(a+b)-2fvw-2gwu-2huv=0 \quad (1)$$

Now if normal to the plane also lie on the cone, we have

$$au^2+bv^2+cw^2+2fvw+2gwu+2huv=0 \quad (2)$$

Adding (1) & (2) we get

$$\boxed{a+b+c=0}$$

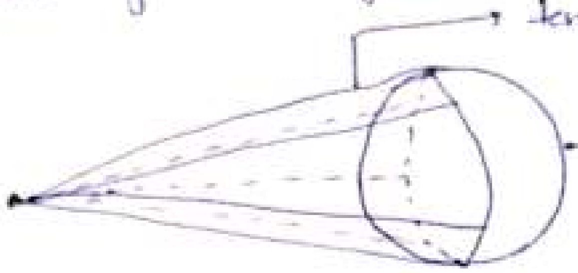
is the required condition ie cone has three mutually $\perp$ generators

Remark. A cone has infinite number of sets of mutually perpendicular generators.

Condition for three mutually $\perp$ tangent planes

$$ab+bc+ca=f^2+g^2+h^2$$

Enveloping Cone → The locus of the tangent lines drawn from a given point to a given surface is called the enveloping cone of the surface.



Equation of the enveloping cone of the sphere.

Let $x^2+y^2+z^2=a^2$ be the sphere and (α,β,γ) be the fixed point (vertex).

Let $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ be any line through $P(\alpha,\beta,\gamma)$

Then the point $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$ must satisfy the eq of sphere. we get a quardatic eq in r. If the line is tangent to the sphere then r has two equal values (ie $b^2-4ac=0$)

ie $(\alpha l+\beta m+\gamma n)^2=(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)$

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ie $(\alpha l+\beta m+\gamma n)^2 = (l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)$

Eliminating l, m, n we get ie $l = x-\alpha$, $m = y-\beta$
$z = n-\gamma$.

$$(\alpha x+\beta y+\gamma z-a)^2 = (x^2+y^2+z^2-a^2)(\alpha^2+\beta^2+\gamma^2-a^2)$$

Ex Find the enveloping cone of the sphere
$x^2+y^2+z^2+2x-2y=2$ with its vertex at $(1,1,1)$.

1. write down the tangent plane for the spher at $(1,1,1)$
$x\cdot1+y\cdot1+z\cdot1+1\cdot(x+1)-1(y+1)-2 = 2x+z-2$

Now by using the formula
$(2x+z-2)^2 = (x^2+y^2+z^2+2x-2y-2)(1+1+1+2-2-2)$
$4x^2+z^2+4+4xz-8x-4z = x^2+y^2+z^2+2x-2y-2$
$3x^2-y^2+4xz-10x+2y-4z+2=0.$