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Normal subgroups and factor groups:-

Normal Subgroup:- The group H is called a normal subgroup or invariant subgroup of G if the left and right cosets of a subgroup H are same for all elements $X \in G$, $X \in H$. Thus

$$XH = HX$$

$$X^{-1}HX = H \quad \forall \; X \in G.$$

Thus every element of XH is equal to some element of HX, or

$$XH_i = H_J X$$

$$\Rightarrow \quad X^{-1}H_J X = H_i$$

Thus H_i is conjugate element of H_J. Thus normal subgroup consists of complete classes of the bigger group. If a subgroup H consists of complete classes of G, then H is a normal subgroup of G.

Example:-1 Write down the normal subgroup of C_{4V}.

Sol:- The normal subgroup of C_{4V} is $K_1 = (E, C_4^2, m_x, m_y)$ $K_2 = (E, C_4^2)$, $K_3 = (C_4, C_4^3)$, $K_4 = (m_x, m_y)$; $K_5 = (\sigma_u, \sigma_v)$

Example: 2 Prove that set K is a group under the given law of composition, where $K = \{(E, C_4^2), (C_4, C_4^3), (m_x, m_y), (\sigma_u, \sigma_v)\}$

Proof:- The set K is defined as $K \equiv (K_1, K_2, K_3, K_4)$

Now we consider the product of K_2, K_3

$$K_2 \cdot K_3 = (C_4, C_4^3)(m_x, m_y) = (\sigma_u, \sigma_v, \sigma_u, \sigma_v) = (\sigma_u, \sigma_v) = K_4$$

Thus $K_4 \in G = K$

Similarly another axiom can be proved. Thus the set K is a group under multiplication law.

<u>factor group:-</u> The group k is called the factor group of G with respect to the normal subgroup (E, C_4^2).

If H is a normal subgroup of G, the set of all the distinct cosets of H in G, together with the coset multiplication is called the factor group or quotient group of G with respect to H and is denoted by

$$k = G/H =$$

order of k is k
order of G is g
order of $H = h$
then $k = g/h$
Thus $g = kh$

<u>Direct product of Groups:</u>

The direct product of two groups, provide
(i) the groups have no common elements except identity E
(ii) each element of H commutes with every element of k.

If $H = \{H_1 = E, H_2, H_3 \dots H_h\}$ of order h and $k = \{k_1 = E, k_2, k_3, \dots k_k\}$ of order k is defined as a group G of order $g = hk$.

$$G = H \otimes k = (E, Ek_2, \dots Ek_k, H_2, H_2k_2 \dots H_hk_k)$$

Thus both H and k are normal subgroup of G.

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Isomorphism and Homomorphism

Isomorphism are relation between two groups with one-to-one correspondence between the elements of two groups G & G'. Thus all groups having simillar multiplication tables having the same structure, then they are said to be isomorphic to each other.

Let $G = \{E, A, B, C, \ldots\}$ and $G' = \{E', A', B', C' \ldots\}$.
then

$$E \leftrightarrow E'; \; A \longleftrightarrow A', \ldots$$

then $AB = C$ in the group G implies that $A'B' = C'$ in the group G'.

Similarly the group $\{1, i, -1, -i\}$ of numbers is isomorphic to the group $\{E, C_4, C_4^2, C_4^3\}$. of rotation under mapping

$$1 \leftrightarrow E, \; i \leftrightarrow C_4, \; -1 \longleftrightarrow C_4^2, \; -i \longleftrightarrow C_4^3.$$

Homomorphism:- If there is many-to-one correspondence or mapping from one group to another, then this type relation between two group is called homomorphism. There is a homomorphism from group G_1 to group G_2 if each elements of G corresponds to unique element $\phi(A)$ of G_2 such that

$$\phi(A.B) = \phi(A)\,\phi(B)$$

where ϕ defines the image relationship between elements of group G to group G'. If n elements of G corresponds to G' one elements, then there is n to 1 mapping

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or homomorphism from G_1 to G_2.

Let $G = \{E, A, B, C \ldots\}$ of ordering
and $G' = \{E_1, E_2 \ldots E_n, A_1 A_2 \ldots A_n \ldots\}$ of ordering ng then suppose that it is possible to split the group G' into sets (E_i), (A_i) etc. each containing n elements such that the the elements of G' can be mapped onto the elements of G according to the scheme

Group G'		Group G
$E_1, E_2 \ldots E_n$	$\longrightarrow$	E
$A_1, A_2, \ldots, A_n$	$\longrightarrow$	A etc

It is called to n-to-1 homomorphism.

The set E_i is a normal subgroup of G'.

The set (E_i) of G' which is mapped onto E of G is called the kernel of homomorphism.

The kernel of homomorphism from G' to G is a normal subgroup of G.

The identity element shows the trivial example of homomorphism. There is a homomorphism from any group G onto the group of order one containing only the identity element is a normal subgroup of any group.

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Permutation Groups:-

Consider a system of n identical object. If any two or more objects are interchanged then the system remains in its original states. Thus it is clear that each interchange of system is invariant. If the system has n objects then there are n! permutations to put objects on the states, hence its order is n! It is known as the permutation group of n objects which is denoted as S_n

For an example; let us consider three identical objects which have the possible permutation which are given as:-

$$E=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad B=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$C=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad D=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad F=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

The lables 1, 2 and 3 stands for three objects and six possible states are

$$\psi_1=(1\ 2\ 3),\quad \psi_2=(2\ 3\ 1),\quad \psi_3=(3.\ 1\ 2)$$

$$\psi_4=(2\ 1\ 3)\quad \psi_5=(3\ 2\ 1),\quad \psi_6=(1\ 3\ 2)$$

The operations are to be interpreted as follows
The operation of A on any state ψ_1 means that the object in position 2 is to be put in position 1, the object in position 3 to be put in position 2. and the object in position is put to position 3.

$$A\psi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} (1\ 2\ 3) = (2\ 3\ 1) = \psi_2$$

$$C\psi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} (2\ 3\ 1) = (3, 2, 1) = \psi_5$$

Now

$$A(C\psi_2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} (3\ 2\ 1) = (2\ 1\ 3) = \psi_4$$

Again

$$F\psi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} (2\ 3\ 1) = (2\ 1\ 3) = \psi_4$$

Therefore $AC\psi_2 = F\psi_2$

Hence it is seen that

$$AC\psi_i = F\psi_i \quad 1 \le i \le 6$$

If a permutation consist of an even number of transposition, it is called an even permutation if it consist of an odd number of transposition it is called an odd permutation.

Here E, A and B are even permutation while C, D and F are odd permutations.

The product of two even or odd permutation is an even permutations. But the product of even permutation with odd permutation is an odd permutation

<u>Example.1:-</u> Prove that a set of a group G is a system of generators of G if and only if no proper subgroup of G exists which contains all the elements of the set S.

Proof:- Let us consider a subset of G such that S is a system of generators of G. Let us assume that there exists a proper subgroup H of G i.e. $S \subseteq H \subseteq G$. Since H is a group and S is contained in H, the powers and products of the elements of S give elements belonging to the group H alone, not G, which contradicts the assumption that S is a system of generators of G. Hence, if S is a system of generators of G, there exists no proper subgroup of G which contains S.

Now assume that there exists no proper subgroup of G which contains S. Let us generate a group by by taking all powers and products of the elements of S. Suppose this gives rise to the group K; i.e $K \subseteq G$. But by assumption, G contains no proper subgroup which contains S. Hence it follows that $K = G$, showing that S is a system of generators of G. Thus if no proper subgroup of G exists which contains S, then S is a system of generators of S.

Hence Proved

Exercise

1 Show that the following sets are group under law of compositions

(i) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ under matrix multiplication

(ii) the set of all complex numbers of unit magnitude under scalar multiplication

2 Generate the matrix group two of whose elements are

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Show that the group is of order 8 and has 5 classes, but is not isomorphic to C_{4v}.