R.H.S

$$m_y \begin{bmatrix} a & b \\ d & c \end{bmatrix} \longrightarrow \begin{bmatrix} b & a \\ c & d \end{bmatrix} \qquad (11)$$

From above discussion it is clear that :-

$$C_4 m_x C_4^{-1} = m_y$$

Hence m_x and m_y are conjuge to each other.

(ii) & (iii) Ca be proved as above methods.

Classes

The above sets of elements are called the conjugacy classes or simply the classes of a group. The identity element E always constitutes a class by itself in any group, since for any element A of the group $A^{-1}EA = E$.

The rotations through angles of different magnitudes must belong to different classes. Thus C_4 and C_4^2 of C_{4V} belong to different classes. Now C_4 and C_4^3 of C_{4V} belong to the same class because of reflection changes the sense of the co-ordinate system

m_x & m_y are reflection of position of square about 5-7 i.e. x axis and 6-8 i.e. y axis so these type of rotation is called same class. Similarly σ_u & σ_v are same class reflection.

m_x & σ_u do not belong to the same class since there is no operation in C_{4v} whi[illegible]n bring to line 1-3 into the line 5-7

This type criteria are useful to obtain the classes of the molecular and the crystallographic point groups simply by inspection.

Multiplication of classes:-

"Let $C_i = (A_1, A_2 \ldots A_m)$ and $C_j = (B_1, B_2, \ldots B_n)$ be two classes which may be same or distinct of a group containing m and n elements respectively. The product or multiplication can be defined as

$$C_i C_j = (A_1 B_1, A_1 B_2, \ldots A_\ell B_k, \ldots A_m B_n)$$

Thus $C_i C_j$ consists of complete classes. Now $A_\ell B_k \in C_i C_j$ and $x, x^{-1} \in C_i C_j$, then

$$x^{-1}(A_\ell B_k)x = (x^{-1} A_\ell x)(x^{-1} B_k x) = A_r B_s$$

where $A_r \in C_i$ and $B_s \in C_j$. Thus $A_r B_s \in C_i C_j$

The product of two classes of a group as a sum of complete classes of the group

$$C_i C_j = \sum_k a_{ijk} C_k .$$

Where a_{ijk} are non-negative integers which gives the number of times the class C_k is contained in the product $C_i C_j$ and sum is over all the classes of the group.

Subgroups:-

A set H is called a subgroup of a group G if H is itself a group under the same law of composition as that of G and and if all the elements of H are also in G.

As an example (E, C_4^2, m_x, m_y), (E, σ_u) are the subgroup of $C_{4v} \equiv (E, C_4, C_4^2, C_4^3, m_x, m_y, \sigma_u, \sigma_v)$

Every group G has two trivial subgroups i.e the identity element and group G itself.

A subgroup H of G is called a proper subgroup if $H \neq G$.

Example:- Prove that the elements belonging to a class in a larger group may not belong to a class in a smaller subgroup.

Proof:- Let us consider the classes of the two subgroups (E, C_4, C_4^2, C_4^3) and (E, C_4^2, m_x, m_y). It is clear that every element constitutes a class by itself in both of this group. The elements C_4 and C_4^3 do not belong to the class in the group (E, C_4, C_4^2, C_4^3) because there is no operation in this group, which changes the sense of the coordinate system similarly m_x and m_y do not belong to the same class in the group (E, C_4^2, m_x, m_y) because there is no operation in this group which can take the x axis onto y axis. Hence the elements belonging to a class in a larger group may not belong to the class in a smaller group.

Cyclic groups:-

A group generated by a single element is a cyclic group. If A is an element of a group G, all integral powers of A such as A^2, A^3 --- must also be in G. If G is a finite group there must exist a finite positive integer n such that

$$A^n = E \quad (1)$$

Where E is identity element. The smallest positive integer satisfying above relation (1) is known as order of the element A. Thus the group $(A, A^2, A^3 \dots A^n \equiv E)$ is a cyclic group.

Cosets:-

Consider a subgroup $H = (H_1 \equiv E, H_2, \dots H_h)$ of order h of a group G which is of order g, let X be any element of G, Construct all the products such as $XE, XH_2, \dots$ etc. and denote the set of these elements by

$$XH = (XE, XH_2, XH_3 \dots XH_h) \quad (1)$$

Now there arise two cases - X may be in the subgroup H or X may not be in H. If

x is a member of H, the set XH must be identical to the group H by the definition of a group.
In the set XH, after rearranging all elements the set H is again obtained, hence it can be written as

$$XH = H \quad \text{if } X \in H$$

On the other hand, if H is a subgroup, X does not belong to H, then it can be shown that no elements of the set XH belongs to H.
Now we take opposite assumption, i.e. suppose that $XH_i \in H \quad \forall \; i (1 \le i \le h)$

Now since H is a group, H_i^{-1} also belongs to H. hence

$$(XH_i)H_i^{-1} = X \in H, \quad \text{contradict}$$

the hypothesis. i.e. $X \notin H$.

This proves that H and XH have no common element, hence

$$XH \cap H = \phi$$

The set XH is called left coset of H in G.
Similarly

$$HX = (EX, H_2X, H_3X \ldots H_hX)$$

if $X \notin H$.

The set HX is called right coset.
here $X; H_i \in G$, but $X \notin H$.

<u>Theorem:-</u> If a group H of order h is a subgroup of group G of order g, then g is an integral multiple of h.

__Proof:-__ Let $H = \{E, H_2, H_3 \dots H_h\}$ be the subgroup of G. Let us consider new element $X \in G$, but $X \notin H$, All the element $XH_i \in G$ $(1 \le i \le h)$. but $XH_i \notin H$, Thus we get new h element of the group G:

$$H \cup XH = (E, H_2, H_3 \dots H_h, X, XH_2 \dots XH_h).$$

If this does not exhaust the group G, then pick up an element Y from the remaining elements of G such that Y belongs neither to H nor to XH. Again form a group YH such that

$$H \cap YH = \phi$$

Now we prove that the set YH and XH are disjoint, If an element YH_i were to be identical to an element XH_J $(1 \le i, J \le h)$. then we have

$$YH_i = XH_J$$

$$Y = XH_J H_i^{-1} \equiv XH_k \text{ say. } H_k \in XH$$

with $1 \le k \le h$

contrary to the hypothesis.

Thus we have h new set of elements of G, making altogether 3h elements

$$H \cup XH \cup YH = (E, H_2 \dots H_h, X, XH_2 \dots XH_h, Y, YH_2 \dots YH_h)$$

If this still does not exhaust the group G, then we pick up one of the remaining elements of G and continue the process. There are every time h new elements are obtained, they must all belong to G. then, order of G is integral multiple of h

Normal subgroups and factor groups:-

Normal Subgroup:- The group H is called a normal Subgroup or invariant subgroup of G if the left and right cosets of subgroup H are same for all elements $X \in G$, $X \in H$. Thus

$$XH = HX$$

$$X^{-1}HX = H \quad \forall\ X \in G.$$

Thus every element of XH is equal to some element of HX, or

$$XH_i = H_J X$$

$$\Rightarrow X H_J X = H_i$$

Thus H_i is conjugate element of H_J. Thus normal subgroup consists of complete classes of the bigger group. If a subgroup H consists of complete classes of G. then H is a normal subgroup of G.

Example:-1. Write down the normal subgroup of C_{4v}.

Sol:- The normal subgroup of C_{4v} is $K_1 = (E, C_4^2, m_x, m_y)$ $K_2(E, C_4^2)$, $K_3 = (C_4, C_4^3)$, $K_4 = (m_x, m_y)$; $K_5 = (\sigma_{u}, \sigma_{v})$

Example:2 Prove that set K is a group under the given law of composition. where $K = \{(E, C_4^2), (C_4, C_4^3), (m_x, m_y), (\sigma_u, \sigma_v)\}$

Proof:- The set K is defined as $K \equiv (K_1, K_2, K_3, K_4)$

Now we consider the product of K_2, K_3

$$K_2 . K_3 = (C_4, C_4^3)(m_x, m_y) = (\sigma_u, \sigma_v, \sigma_u, \sigma_v) = (\sigma_u, \sigma_v) = K_4$$

Thus $K_4 \in G = K$

Similarly another axium can be proved. Thus the set K is a group under multiplication law.