from those of the second set by a similarity transformation of the cordinate vectors of the vector space in which both the representations are defined. This canbe represented in short

$$T_1 = S^{-1} T_2 S$$

If two representations of a group are not equivalent to eachother, they are called to be in equivalent or distinct representations

Invariant Subspaces and Reducible representations:

For every element $A \in G$, and every vector $\phi \in L_n$ then $A\phi \in L_n$, thus L_n is closed under the transformation of G. It means that the operation of any element of G on any vector of L_n doesnot take outside L_n.

A vector space L_m is said to be a subspace of L_n if every vector of L_m is also contained in L_n. L_m is called a proper subspace of L_n if the vectors of L_m do not exhaust the L_n. Thus L_n is also a subspace itself.

The vector space L_n may posses a proper subspace L_m which is also invariant under G. Then L_m is called invariant under G. and the space L_n is said to be reducible under G.

Reducibility of representation:

Let $\{T(E), T(A), T(B) \ldots\}$ be a representation of G

in L_n. and L_n has an invariant subspace L_m under G. then T(A) canbe written as

$$T(A) = \left[\begin{array}{c|c} D^1(A) & 0 \\ \hline X(A) & D^2(A) \end{array}\right] \qquad (1)$$

where $D^1(A)$ and $D^2(A)$ are square matrices of order m and n-m respectively.
$X(A)$ is of order $(n-m) \times m$
0 is a null matrix of order $m \times (n-m)$.
To show this, we use the row vector notation for the vectors

$$\phi_i = (0,0,0 \ldots 1_i\ 0\,0\,0 \ldots)$$

here i^{th} column has unity and other elements are zero. The operation of $A \in G$ on a basis vector ϕ_μ $(1 \leq \mu \leq m)$ is given by

$$A\phi_\mu = (0\,0\,0 \ldots 1_\mu 0 \ldots 0) \left[\begin{array}{ccc|ccc} T_{11} & \ldots & T_{1m} & T_{1,m+1} & \ldots & T_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ T_{m1} & \ldots & T_{mm} & T_{m,m+1} & \ldots & T_{mn} \\ \hline T_{m+1,1} & \ldots & & & \ldots & T_{m+1,n} \\ \vdots & & \vdots & & & \vdots \\ T_{n1} & \ldots & T_{nm} & T_{n,m+1} & \ldots & T_{nn} \end{array}\right]$$

$$= (T_{\mu 1}, T_{\mu 2} \ldots T_{\mu m}, T_{\mu, m+1} \ldots \ldots T_{\mu n})$$

Since L_m is itself invariant under G, transformed vector $A\phi_\mu$ is also belongs to L_m; hence its components along the basis vectors $\phi_{m+1}, \phi_{m+2} \ldots \phi_n$ must be zero. i.e.

$$T_{\mu k}(A) = 0 \qquad m+1 \leq k \leq n$$

Hence $T(A)$ has the form as shown in eq.(1).

Let us consider $A, B \in G$ then $AB = C$, In terms of the matrices of representation considered as $T(A) \cdot T(B) = T(C)$

$$T(C) = \left[\begin{array}{c|c} D^{(1)}(A) & 0 \\ \hline X(A) & D^{(2)}(A) \end{array}\right]\left[\begin{array}{c|c} D^{(1)}(B) & 0 \\ \hline X(B) & D^{(2)}(B) \end{array}\right]$$

$$= \left[\begin{array}{c|c} D^{1}(A) \cdot D^{(1)}(B) & 0 \\ \hline X(A)\, D^{1}(B) + D^{2}(A)\, X(B) & D^{2}(A)\, D^{2}(B) \end{array}\right]$$

$$= \left[\begin{array}{c|c} D^{1}(C) & 0 \\ \hline X(C) & D^{2}(C) \end{array}\right]$$

Therefore we have

$$D^{1}(C) = D^{1}(A)\, D^{(1)}(B)$$

$$D^{2}(C) = D^{(2)}(A)\, D^{(2)}(B)$$

$$X(C) = X(A)\, D^{1}(B) + D^{2}(A)\, X(B)$$

Thus it is clear that $D^{(1)} = \{D^{1}(E), D^{(1)}(A) \ldots\}$

$D^{(2)} = \{D^{(2)}(I), D^{(2)}(A) \ldots\}$. also give two new representations of dimensions m and n-m respectively for the group G. Thus $\{\phi_1, \phi_2, \ldots \phi_m\}$ are the basis vectors for the representation $D^{(1)}$ and remaining (n-m) basis vectors $\{\phi_{m+1}, \ldots \phi_n\}$ for $D^{(2)}$.

Thus T is said to be a reducible representation. Thus the reducibility of a representation is connected with the existence of a proper invariant subspace of the full space.

Example 1:- Prove that any representation T of finite group, whose matrices may be non-unitary, is equivalent to a representation by unitary matrices

Proof:- We define a hermitian matrix to prove above theorem

$$H = \sum_{A \in G} T(A) T^{\dagger}(A) \quad (1)$$

The hermitian matrix can be fully diagonilised by a unitary transformation U., then

$$U^{-1} H U = H_d \quad (2)$$

Using eq.(1) & eq.(2), we get

$$H_d = U^{-1} \sum_{A \in G} T(A) T^{+}(A) U$$

$$= U^{-1} \sum_{A \in G} T(A) U U^{-1} T^{+}(A) U$$

$$= \left[U^{-1} \sum_{A \in G} T(A) U \right] \left[U^{-1} \sum_{A \in G} T^{+}(A) U \right]$$

$$= \sum_{A \in G} T'(A) T'^{+}(A) \qquad -(3)$$

where $T'(A) = U^{-1} T(A) U$, then K^{th} diagonal element of eqn (3), we get

$$[H_d]_{kk} = d_k = \sum_{A \in G} \sum_{j} T'_{kj}(A) . T'^{+}_{jk}(A)$$

$$= \sum_{A \in G} \sum_{j} T'_{kj}(A) T'^{+}_{kj}(A)$$

$$= \sum_{A \in G} \sum_{j} | T'_{kj}(A) |^2 \qquad -(4)$$

The eqn (4) shows that $d_k \geq 0$. But it can be zero if and only if $T'_{kJ}(A) = 0 \ \forall$ J and $A \in G$. Hence $d_k > 0$, must be positive.

It is also clear that H_d is a non-singular matrix, so that any power of matrix H_d by taking the corresponding power of all the diagonal elements of H_d i.e.

$$[(H_d)^p]_{kk} = |d_k|^p \qquad (5)$$

where p is any real number, positive or negative.

The similarity transformation matrix which converts the non-unitary matrices $T(A)$ into unitary matrices $\Gamma(A)$ is obtained as

$$V = U H_d^{1/2} \quad \text{giving} \quad (6)$$

$$\Gamma(A) = V^{-1} T(A) V$$
$$= H_d^{-1/2} U^{-1} T(A) U H_d^{1/2}$$
$$= H_d^{-1/2} T'(A) H_d^{1/2} \quad (7)$$

To varify that the matrices $\Gamma(A)$ are indeed unitary, so

$$\Gamma(A) \Gamma^{+}(A) = [H_d^{-1/2} T'(A) H_d^{1/2}][H_d^{1/2} T'^{+}(A) H_d^{-1/2}]$$
$$= H_d^{-1/2} T'(A) H_d T'^{+}(A) H_d^{-1/2}$$
$$= H_d^{-1/2} T'(A) \sum_{B \in G} T'(B) T'^{+}(B) T'^{+}(A) H_d^{-1/2}$$
$$= H_d^{-1/2} \sum_{B \in G} T'(AB) T'^{+}(AB) H_d^{-1/2}$$
$$= H_d^{-1/2} H_d H_d^{-1/2} \quad \text{from eq (3)}$$
$$= E$$

Hence $\Gamma(A)$ is a unitary matrix.

If the elements of the group G are unitary operators the similarity transformation of representation T to the representation Γ has a simple physical meaning that it implies the oblique system of coordinate axes are orthogonal to eachother. The non unitary nature of matrices are not orthogonal on the basis vector L_n.

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Irreducible representations:-

The process of reducible representation can be carried on until we can find no unitary transformation which reduces all the matrices of a representation further. After reducing the final form of the matrices of the representation Γ can be written as

$$\Gamma(A) = \begin{bmatrix} \Gamma^{(1)}(A) & & & 0 \\ & \Gamma^{(2)}(A) & & \\ & & \ddots & \\ 0 & & & \Gamma^{(s)}(A) \end{bmatrix} \quad \text{etc}$$

with all the matrices of Γ having the same reduced structure. When such a complete reduction of a representation is achieved, the component representations $\Gamma^{(1)}, \Gamma^{(2)}$ ---- $\Gamma^{(s)}$ are called the irreducible representation of the group G and the representation Γ is said to be fully reduced.

An irreducible representation may occur more than once in the reduction of a reducible Γ. The matrices of the representation Γ are just the direct sum of the matrices of the component irreducible representations and this may be denoted by

$$\Gamma = a_1\Gamma^{(1)} \oplus a_2\Gamma^{(2)} \oplus \ldots \oplus a_l\Gamma^{(l)}$$
$$= \sum_i a_i\Gamma^{(i)}$$

The symbole is denoted as direct sum of Γ.

Schur's Lemma 1:-

If $\Gamma^{(i)}$ is an irreducible representation of a group G and if a matrix P commutes with all the matrices of $\Gamma^{(i)}$, then P must be constant, i.e. $P = cE$, where c is a scalar

Proof:- Let A be any element of the group G, then it is given that

$$\Gamma^{(i)}(A)P = P\Gamma^{(i)}(A) \quad \forall A \in G, \quad (1)$$

If the dimension of $\Gamma^{(i)}$ is n, P is square matrix of order n. and taken to be unitary and each of the matrices $\Gamma(A), \Gamma(B)$ etc., possess a complete set of n eigen vectors, P also has n linearly independent eigenvectors. x_j (say) with eigenvalues c_j, then we have

$$P x_j = c_j x_j \quad (2)$$

Multiplying bothsides from the left by $\Gamma^{(i)}(A)$ of eq(2), we get

$$\Gamma^{(i)}(A) P_{x_j} = \Gamma^{(i)}(A) c_j x_j$$

$$P\Gamma^{(i)}(A) x_j = c_j \Gamma^{(i)}(A) x_j \quad -(3)$$

This shows that $\Gamma^{(i)}(A)x_j$, $\forall A \in G$, are eigenvector of P with the same eigenvalue c_J. Let there be m such independent eigenvectors of P having the same eigenvalue c_J. But eigenvectors belonging to an eigenvalue generator subspace L_m, which is invariant under G. Now if L_m is a proper subspace of L_n, where L_m is not the same as L_n, then L_n is also invariant subspace. The representation $\Gamma^{(i)}$ must be reducible, which is contrary to the hypothesis. Therefore L_m must be identical with L_n, making all the eigenvalues of P each to equal to each other, say $c_j \equiv c$, giving $P = cE$.

Thus invariant subspace L_m may contain only the null vector. However, this case excluded from consideration because if x is a null vector, it trivially satisfies the eigenvalue equal to $Px = cx$ with an arbitrary eigen value c.

Hence the theorem is proved

Schur's Lemma 2:-

If $\Gamma^{(i)}$ and $\Gamma^{(J)}$ are two irreducible representations of dimensions l_i and l_j respectively of a group G and if a matrix M (of order $l_i \times l_j$) satisfies the relation

$$\Gamma^{(i)}(A)\, M = M\, \Gamma^{(J)}(A) \qquad \forall\ A \in G \qquad (1)$$

then either (a) $M=0$, the null matrix or (b) $\det M \neq 0$, in which case $\Gamma^{(i)}$ and $\Gamma^{(j)}$ are equivalent representations.

Proof:- Two representation can be equivalent only if their dimensions are equal. Hence if $l_i \neq l_j$ only case (a) applies.

Taking the hermitian conjugate of both sides of eq (1), we have

$$M^{+}\Gamma^{(i)+}(A) = \Gamma^{(j)+}M^{+} \qquad \forall A \in G$$

or $$M^{+}\Gamma^{(i)}(A^{-1}) = \Gamma^{(j)}(A^{-1})M^{+} \qquad \forall A \in G$$

Multiplying from the right by M, we get

$$M^{+}M\,\Gamma^{(j)}(A^{-1}) = \Gamma^{(j)}(A^{-1})M^{+}M \qquad \forall A \in G$$

From lemma 1,

$$M^{+}M = cE \qquad (2)$$

and $l_i = l_j = n$

then $\det M^{+}M = c^{n}$

if $c \neq 0$, then $\det M \neq 0$, so

$$\Gamma^{(j)}(A) = M^{-1}\Gamma^{(i)}(A)M \qquad \forall A \in G.$$

$\Gamma^{(j)}$ & $\Gamma^{(i)}$ are equivalent representations

if $c = 0$, then $(i,i)^{th}$ element of eq (2) we get

$$\sum_{k} M^{+}_{ik}M_{ki} = 0$$

or $$\sum_{k} M^{*}_{ki}M_{ki} = \sum_{k}|M_{ki}|^{2} = 0$$

which is possible if only if $M_{ki} = 0$.
for $1 \le k \le n$, i is arbitrary $1 \le i \le n$.
hence $M = 0$.

In second case, when $l_i \ne l_J$, then $l_i < l_j$
we supplement the matrix M by writing $(l_j - l_i)$ rows of zeros to give a new matrix M'

$$M' = \begin{array}{c} \leftarrow l_j \rightarrow \\ \left[\begin{array}{c} M \\ \hline 0 \end{array} \right] \end{array} \begin{array}{l} \} \, l_i \\ \} \, l_J - l_i \end{array}$$

$$\begin{array}{cc} \leftarrow l_i \longrightarrow & \leftarrow l_J - l_i \rightarrow \end{array}$$

$$M'^{+} = [\; M^{+} \; \vdots \; 0 \;] \; l_J$$

It can be seen that $M'^{+} M' = M^{+} M$
and hence

$$\det(M'^{+} M') = \det(M' M)$$

$$\det(M'^{+}) \det(M') = c^n$$

Now put $l_J = n$.

Now $\det(M') = \det(M'^{+}) = 0$,

$$c = 0$$

hence $M^{+} M = 0$. once again taking the (i,i) element of $M^{+} M$, we get $M = 0$

Hence theorem is proved.

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The orthogonality theorem:-

Let us construct a matrix M given by

$$M = \sum_{A \in G} \Gamma^{(i)}(A) X \Gamma^{(j)}(A^{-1}) \quad (1)$$

Multiplying by $\Gamma^{(i)}(B)$

$$\Gamma^{(i)}(B) M = \Gamma^{(i)}(B) \sum_{A \in G} \Gamma^{(i)}(A) X \Gamma^{(j)}(A^{-1})$$

$$= \sum_{C \in G} \Gamma^{(i)}(C) X \Gamma^{(j)}(C^{-1}) \Gamma^{(j)}(B)$$

$$= M \Gamma^{(j)}(B) . \quad \text{where } BA = C$$

by second lemma of schur ; $M = 0$

Taking (k,s) element of eq (1)

$$\sum_{A \in G} \sum_{pq} \Gamma^{(i)}_{kp}(A) X_{pq} \Gamma^{(j)}_{qs}(A^{-1}) = 0$$

If $X_{pq} = \delta_{pm} \delta_{qn}$,

$$\sum_{A \in G} \Gamma^{(i)}_{km}(A) \Gamma^{(j)}_{ns}(A^{-1}) = 0$$

$$\sum_{A \in G} \Gamma^{(i)}_{km}(A) \Gamma^{(j)*}(A) = 0$$

Next construct $N = \sum_{A \in G} \Gamma^{(i)}(A) X \Gamma^{(i)}(A^{-1})$

then $\Gamma^{(i)}(A) N = N \Gamma^{(i)}(A) \quad \forall A \in G$