thus $\Gamma^{(h)}(H_i)\Gamma^{(h)}(H_m) = \Gamma^{(h)}(H_p)$

$\Gamma^{(g)}(G_j)\Gamma^{(g)}(G_n) = \Gamma^{(g)}(G_q)$

The direct product of the elements matrices, on the respective sides of the equation, we have

$$\Gamma^{(h)}(H_p)\otimes\Gamma^{(g)}(G_q) = [\Gamma^{(h)}(H_i)\Gamma^{(h)}(H_m)]\otimes[\Gamma^{(g)}(G_j)\Gamma^{(g)}(G_n)]$$

$$= [\Gamma^{(h)}(H_i)\otimes\Gamma^{(g)}(G_j)][\Gamma^{(h)}(H_m)\otimes\Gamma^{(g)}(G_n)] \quad -(2)$$

If we define new matrix

$$\Gamma^{(k)}(K_{pq}) = \Gamma^{(h)}(H_p)\otimes\Gamma^{(g)}(G_q) \quad -(3)$$

Then eq (2) becomes

$$\Gamma^{(k)}(K_{pq}) = \Gamma^{(k)}(K_{ij})\,\Gamma^{(k)}(K_{mn})$$

Thus the direct product of representations of two commuting groups is a representations of the direct product group

If $\Gamma^{(h)}$ and $\Gamma^{(g)}$ are IR of H and G, then $\Gamma^{(k)} = \Gamma^{(h)}\otimes\Gamma^{(g)}$ is an IR of K. Thus irreducibility gives

$$\left.\begin{aligned} &\sum_{H_i\in H}\chi^{(h)}(H_i)\,\chi^{(h)*}(H_i) = h \\ &\sum_{G_i\in G}\chi^{(g)}(G_i)\,\chi^{(g)*}(G_i) = g \end{aligned}\right\} \quad -(4)$$

Taking the product of the respective sides we get

$$h g \equiv k = \sum_{H_i \in H} \sum_{G_j \in G} [\chi^{(h)}(H_i)\chi^{(g)}(G_j)][\chi^{(h)*}(H_i)\chi^{(g)*}(G_j)]$$

The characters of K is product of character of H and G, Hence, the above equation becomes as

$$k = \sum_{K_{ij} \in K} \chi^{(k)}(K_{ij})\chi^{(k)*}(K_{ij}).$$

Hence this shows that if $\Gamma^{(h)}$ and $\Gamma^{(g)}$ are reducible representations then $\Gamma^{(k)}$ is also reducible representations.

Example:- Prove that all IR of K are the direct products of an IR of H and one of G.

Proof:- Let the number of IR of H be c_h and their dimensions $l_i^{(h)}$ $(1 \le i \le c_h)$. Let also the number of IR of G be c_g $(1 \le j \le c_g)$ of dimension $l_j^{(g)}$ then

$$\sum_{i=1}^{c_h} |l_i^{(h)}|^2 = h$$

$$\sum_{j=1}^{c_g} |l_j^{(g)}|^2 = g$$

The irreducible representations of K is given as

$$l_{ij}^{(k)} = l_i^{(h)} l_j^{(g)}.$$

Consider now the sum of the squares of dimensions of the IR of K is obtained as

$$\sum_{i=1}^{c_h} \sum_{j=1}^{c_g} |l_{ij}^{(k)}|^2 = \sum_{i=1}^{c_h} \sum_{j=1}^{c_g} |l_i^{(h)}|^2 |l_j^{(g)}|^2 = hg = k$$

or

$$\sum_{n=1}^{c_h c_g} |l_n^{(k)}|^2 = k \qquad \text{--- (1)}$$

Thus the above equation shows that the direct products of the irreducible representations of H and G exhaust all the irreducible representation of K i.e. there is no IR of K which cannot be expressed as a direct product of an IR of H and one of G. Now if we denote the number of the IR of K by c_k, then

$$c_k = c_h c_g$$

Hence Proved

Basis functions for representations of the direct product group:-

The basis function for $\Gamma^{(k)}$ of K can be constructed by direct product of $\Gamma^{(h)}$ and $\Gamma^{(g)}$ which is denoted as $\Gamma^{(k)}$. Thus we get direct product of $\Gamma^{(h)}$ and $\Gamma^{(g)}$

For this $\Gamma^{(k)} = \Gamma^{(h)} \otimes \Gamma^{(g)}$

If $l^{(h)} \equiv a$ for $\Gamma^{(h)} = \{\phi_1, \phi_2, \ldots \phi_a\}$
and $l^{(g)} \equiv b$ for $\Gamma^{(g)} = \{\chi_1, \chi_2 \ldots \chi_b\}$.

Then $\Gamma^{(k)} = \Gamma^{(h)} \otimes \Gamma^{(g)}$ has ab basis function $\psi_{mn} = \phi_m \chi_n$ and

$1 \leq m \leq a$, $1 \leq n \leq b$. if K is denoted as $K_{pq} = H_p G_q$, then

$$K_{pq}\psi_{mn} = \sum_{(kl)=1}^{ab} \psi_{kl} \Gamma^{(k)}_{kl,mn}(K_{pq})$$

$$= \sum_{(kl)=1}^{ab} \phi_k \chi_l \left[\Gamma^{(h)}_{km}(H_p)\Gamma^{(g)}_{ln}(G_q)\right]$$

$$= \left[\sum_{k=1}^{a} \phi_k \Gamma^{(h)}_{km}(H_p)\right]\left[\sum_{l=1}^{b} \chi_l \Gamma^{(g)}_{ln}(G_q)\right]$$

$$= (H_p \phi_m)(G_q \chi_n)$$

Thus the operators of the two constituent groups act on functions of their respective Hilbert spaces only

Let us consider two groups of order 2 as $H = \{E_x, m_x\}$ and $G = \{E_y, m_y\}$. Since

m_x, commutes with m_y we can take the direct product of H and G for order four, with elements $E = E_x E_y$, $A = E_x m_y$, $B = m_x E_y$, $C = m_x m_y$. The group H and G's IR are given below

Group H

	E_x	m_x
$\Gamma_1^{(h)}$	1	1
$\Gamma_2^{(h)}$	1	-1

Group G

	E_y	m_y
$\Gamma_1^{(g)}$	1	1
$\Gamma_2^{(g)}$	1	-1

IR of K can be obtained by taking direct product of IR of H and G. These are given below

Group K

	E	A	B	C
$\Gamma_1^{(k)} \equiv \Gamma_{11}^{(k)}$	1	1	1	1
$\Gamma_2^{(k)} \equiv \Gamma_{12}^{(k)}$	1	-1	1	-1
$\Gamma_3^{(k)} \equiv \Gamma_{21}^{K}$	1	1	-1	-1
$\Gamma_4^{k} \equiv \Gamma_{22}^{(k)}$	1	-1	-1	1

Let ϕ_1, ϕ_2 as basis function of two IR of H and χ_1, χ_2 are two basis function of G.

$E_x \phi_1 = \phi_1$, $m_x \phi_1 = \phi_1$, $E_x \phi_2 = \phi_2$, $m_x \phi_2 = -\phi_2$

$E_y \chi_1 = \chi_1$, $m_y \chi_1 = \chi_1$, $E_x \chi_2 = \chi_2$, $m_y \chi_2 = \chi_2$

As IR of $\Gamma_n^{(k)} = \Gamma_{ij}^{(k)}$ of k, then basis function is

$$\chi_{ij} \equiv \phi_i \chi_j \qquad i, j = 1, 2$$

IR of $\Gamma_2^{(k)} \equiv \Gamma_{12}^{(k)}$ has the basis function $\chi_{12} \equiv \phi_1 \chi_2$ can be written as

$$E \chi_{12} = (E_x \phi_1)(E_y \chi_2) = \phi_1 \chi_2 = \chi_{12}$$
$$A \chi_{12} = (E_x \phi_1)(m_y \chi_2) = \phi_1(-\chi_2) = -\chi_{12}$$
$$B \chi_{12} = (m_x \phi_1)(E_y \chi_2) = \phi_1 \chi_2 = \chi_{12}$$
$$C \chi_{12} = (m_x \phi_1)(m_y \chi_2) = \phi(-\chi_2)_1 = -\chi_{12}$$

If there are two distinguishable particle (such as electron and proton) whose wave functions transform according to some representations of two different symmetry groups, then the wave function of the system as a whole will transform according to the representations of the direct product group.

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