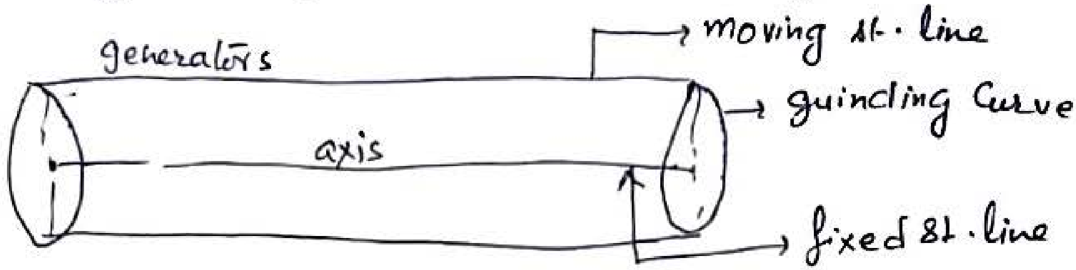
Sem II 4Y - Coordinate Geometry

Topic- Cylinder

Def. A surface generated by a st. line which is parallel to a fixed st. line and intersects a given curve or touches a given surface is called a cylinder

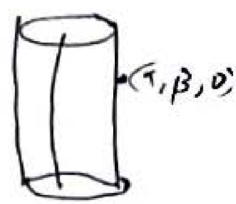


Equation of a Cylinder

Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be the st line parallel to the generator of the cylinder which intersect the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad z = 0 \quad \text{—— (1)}$$

To find the equation of Cylinder.



Let (α, β, γ) be any pt on the cylinder

eq of the generator $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

This meets the plane $z=0$ ~~in the point~~

ie $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = -\frac{\gamma}{n}$

is the point $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

∵ generator intersect the given conic (1) then point must satisfy (1) ie $a\left(\alpha - \frac{l\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{l\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + b\left(\beta - \frac{m\gamma}{n}\right)^2$

$+ 2g\left(\alpha - \frac{l\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0$

Now locus of (α, β, γ) is the eq of cylinder

ie $a(nx - lz)^2 + 2h(xn - lz)(yn - mz) + b(yn - mz)^2$

$+ 2g(nx - lz) + 2f(ny - mz) + cn^2 = 0.$

Cases →

1. If generators are parellel to x-axis then eq of the guiding curve is $f(z, y) = 0,\ x = 0$
2. If generators are $\parallel$ to y-axis then eq of the guinding curve is $f(x, z) = 0,\ y = 0$

Ex Find the equation of a cylinder whose generators are $\parallel$ to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and passes through the curve $x^2 + 2y^2 = 1,\ z = 0$.

Sol. Let (α, β, γ) be the pt on the cylinder
the generator of the cylinder is

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{z-\gamma}{3} \quad \text{—— (1)}$$

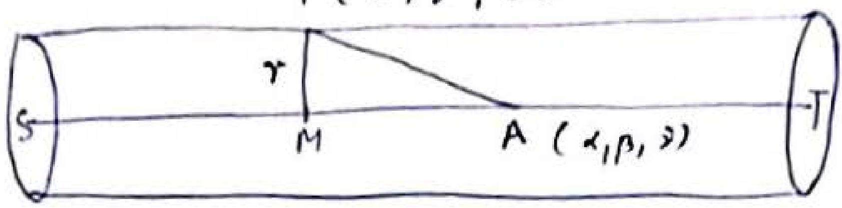
∵ it meet at $z = 0$ then put $z = 0$ in (1)
we get a point $\left(\alpha - \frac{\gamma}{3},\ \beta + \frac{2\gamma}{3},\ 0\right)$ which meets the given curve ie

$$\left(\alpha - \frac{\gamma}{3}\right)^2 + 2\left(\beta + \frac{2\gamma}{3}\right)^2 = 1$$

locus of (α, β, γ) is $(3x - z)^2 + 2(3y + 2z)^2 = 9$

Right Circular Cylinder (R.C.C.)

Def - A RCC is the surface generated by the st. line which is at the constant distance from a fixed st. line which and parallel to it



r is the constant distance (radius of the cylinder)
ST is a fixed line is axis of the cylinder.

To find the equation of RCC.

Let r be the radius and eq of the axis is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

Let $P(x', y', z')$ be any pt on the cylinder
the according to fig we get.

$$PM^2 = AP^2 - AM^2$$

$$r^2 = \sum (x'-\alpha)^2 - \sum \left\{ \frac{l(x'-\alpha)}{\sqrt{l^2+m^2+n^2}} \right\}^2$$

where $\sum (x'-\alpha)^2 = (x'-\alpha)^2 + (y'-\beta)^2 + (z'-\gamma)^2$

Hence locus of (x', y', z') is

$$r^2 (l^2+m^2+n^2) = (l^2+m^2+n^2) \sum (x-\alpha)^2 - \sum (l(x-\alpha))^2$$

Corollary 1. eq of RCC is

$$(l^2+m^2+n^2)(x^2+y^2+z^2-r^2) = (lx+my+nz)^2$$

? if eq of the axis is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

Coro II The equation of RCC whose axis is z axis is $x^2 + y^2 = r^2$

i.e eq of z axis is $\frac{0}{l} = \frac{0}{m} = \frac{z}{n}$

ex Find the equation of a RCC of radius 2 whose axis passes through the point (1, 2, 3) and has direction Cosines proportional to 2, −3, 6.

Sol. Let (α, β, γ) be the pt on the cylinder

eq of the axis is $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-3}{6}$

radius is 2

∴ by using the formula $PM^2 = AP^2 - AM^2$

$$(2)^2 = \sum (\alpha - 1)^2 + \left(\frac{2(\alpha-1) - 3(\beta-2) + 6(\gamma-3)}{\sqrt{4+9+36}}\right)^2$$

$$4 \times 49 = (\alpha-1)^2 + (\beta-2)^2 + (\gamma-3)^2 + [2(\alpha-1) - 3(\beta-2) + 6(\gamma-3)]^2$$

locus of (α, β, γ) is

$$4 \times 49 = (x-1)^2 + (y-2)^2 + (z-3)^2 + [2(x-1) - 3(y-2) + 6(z-3)]^2$$

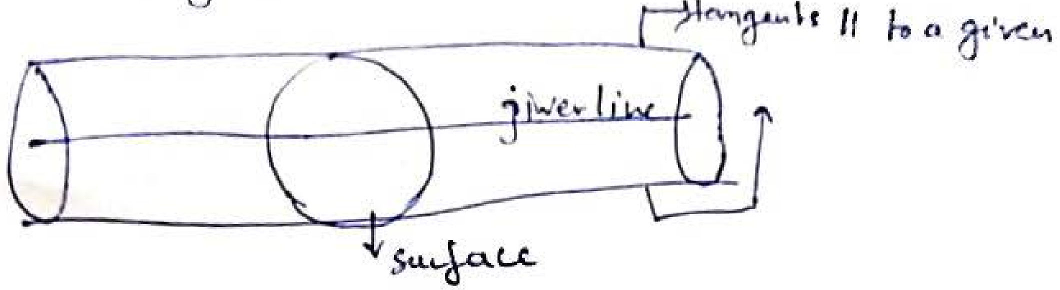
After solving the above equation we get

$$45x^2 + 40y^2 + 13z^2 + 36yz - 24zx + 12xy - 42x - 280y - 126z + 294 = 0.$$

Enveloping Cylinder

Def. A cylinder whose generators are $\parallel$ to a fixed line and touch a surface.

or It is locus of tangents drawn to a given surface and $\parallel$ to a given line.



To find the equation of enveloping cylinder.

Let $x^2+y^2+z^2=a^2$ be the given surface and let-

$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be the line $\parallel$ to the generators

Let $P(\alpha, \beta, \gamma)$ be any point on the locus.

Any line through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ (say)} \quad \text{—} (1)$$

Any point Q on this line is $(\alpha+lr, \beta+mr, \gamma+nr)$

This point will lie on the surface i.e

$$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2=a^2$$

i.e $r^2(l^2+m^2+n^2)+2r(\alpha l+\beta m+\gamma n)+\alpha^2+\beta^2+\gamma^2-a^2=0$.

r has two values which are common as (1) is the tangent to the given surface therefore using $b^2-4ac=0$

i.e $(\alpha l+\beta m+\gamma n)^2=(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)$

Hence locus of (α, β, γ) is

$$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2-a^2)$$

Q. Find the equation of Enveloping Cylinder of the sphere $x^2+y^2+z^2-2x+4y=1$ having its generators $\parallel$ to the line $x=y=z$. Also find its guiding curve.

Sol. eq of the line $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$ — (1)

Let (α,β,ν) be the pt of on the cylinder then equation of generator $\parallel$ to the line (1) is

$$\frac{x-\alpha}{1}=\frac{y-\beta}{1}=\frac{z-\nu}{1}=r$$

The pt of $(r+\alpha,\ \beta+r,\ \nu+r)$ must lie on the given sphere

$$\therefore\ (\alpha+r)^2+(\beta+r)^2+(\nu+r)^2-2(\alpha+r)+4(\beta+r)=1$$

$$3r^2+2(\alpha+\beta+\nu-2+4)r+\alpha^2+\beta^2+\nu^2-2\alpha+4\beta-1=0$$

Now using $b^2-4ac=0$ as r has equal roots

$$(\alpha+\beta+\nu+2)^2=(\alpha^2+\beta^2+\nu^2-2\alpha+4\beta-1)(3)$$

$$\alpha^2+\beta^2+\nu^2+4+2\alpha\beta+2\beta\nu+2\nu\alpha+4\alpha+4\nu+4\beta$$
$$=3\alpha^2+3\beta^2+3\nu^2-6\alpha+12\beta-3$$

$$2\alpha^2+2\beta^2+2\nu^2-2\alpha\beta-2\beta\nu-2\nu\alpha-10\alpha+8\beta-4\nu-10=0$$

ie $\alpha^2+\beta^2+\nu^2-\alpha\beta-\beta\nu-\nu\alpha-5\alpha+4\beta-2\nu-5=0$

Hence locus of (α,β,ν) is

$$x^2+y^2+z^2-xy-yz-zx-5x+4y-2z-5=0$$