Neutron - Proton Scattering

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(Low energy)

The relative motion of two particles having masses M_1 and M_2 can be described by the wave equation given by

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V(r)\psi = E\psi$$

where μ is reduced mass and E is the internal energy of the system $\boxed{E = E_L - E_c}$

E_L — energy in the lab system
E_c — Kinetic energy of the centre of mass

$$E_c = \frac{M_1}{M_1+M_2}E_L$$

for n-p scattering $M_1 = M_2 = M$ (say)
so that
$$E_c = \frac{E_L}{2}$$

So only half the lab energy is available for scattering in the centre of mass system

Relation between θ_c & θ_L

$$\theta_c = 2\theta_L$$

where θ_L is the angle of scattering in L system and θ_c is the angle of scattering in COM system.

Now the wave equation can be written as

$$\nabla^2\psi + \frac{M}{\hbar^2}\left[E - V(r)\right]\psi = 0$$

where $\psi = \psi(r, \theta, \phi)$; θ and ϕ are the COM angles, r is the distance between neutron and proton, $E > 0$.

Method of Partial waves: for large r values.

$$\psi_{inc} = \frac{1}{2iKr}\sum_{l=0}^{\infty}(2l+1)i^l\left[\overbrace{\mathrm{Exp}\, i\left(Kr - l\frac{\pi}{2}\right)}^{\text{outgoing wave}} - \underbrace{\mathrm{Exp}\, {-i}\left(Kr - l\frac{\pi}{2}\right)}_{\text{Incoming wave}}\right\} P_l(\cos\theta)$$

$P_l(\cos\theta)$ is Legendre polynomial of order l

and $K^2 = \frac{2\mu E}{\hbar^2} = \frac{ME}{\hbar^2}$, where $\mu = \frac{M}{2}$

when scatterer is present, the sph outgoing wave is affected either in Phase or in Amplitude or both. If elastic scattering is taking place (no reaction), then only phase is affected.

Total wave function when scatterer is present

$$\psi(r) = \frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)i^l\left[\eta_l\, e^{i(kr - l\frac{\pi}{2})} - e^{-i(kr - l\frac{\pi}{2})}\right] P_l(\cos\theta)$$

$$= \frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)i^l\left[\eta_l\, e^{i(kr - l\frac{\pi}{2})} + e^{i(kr - l\frac{\pi}{2})} - e^{-i(kr - l\frac{\pi}{2})} - e^{-i(kr - l\frac{\pi}{2})}\right] P_l(\cos\theta)$$

$$= \frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)i^l\left[\eta_l\, e^{i(kr - l\frac{\pi}{2})} - e^{-i(kr - l\frac{\pi}{2})}\right] P_l(\cos\theta)$$

$$+ \frac{1}{2ikr}\sum_{l}(2l+1)i^l\left[e^{i(kr - l\frac{\pi}{2})} - e^{-i(kr - l\frac{\pi}{2})}\right] P_l(\cos\theta)$$

$$= \psi_{inc} + \psi_{sc}$$

where $\psi_{sc} = \frac{1}{2ikr}\sum(2l+1)i^l\left[\eta_l\, e^{i(kr - l\frac{\pi}{2})} - e^{+i(kr - l\frac{\pi}{2})}\right] P_l(\cos\theta)$

$$\psi_{sc} = \frac{1}{r} f(\theta) e^{ikr}$$

$$\psi_{sc} = \frac{1}{r} \Sigma \frac{(\eta_l - 1)(2l+1)}{2ik} i^l e^{-il\frac{\pi}{2}} P_l(\cos\theta) e^{ikr}$$

$$f(\theta) = \frac{1}{2ik} \Sigma (2l+1) \{ e^{2i\delta_l} - 1 \} P_l(\cos\theta)$$

$$= \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta)$$

$$\sin\delta_l = \frac{e^{i\delta_l} - e^{-i\delta_l}}{2i}$$

So the differential cross section will be

$$\sigma(\theta) = |f(\theta)|^2$$

$$= \frac{1}{K^2}\left|\sum_{l=0}^{\infty}(2l+1)\, e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)\right|^2$$

$$\sigma_{total} = \int \sigma(\theta)\, d\Omega = \int \sigma(\theta)\, 2\pi \sin\theta\, d\theta$$

$$\sigma = \frac{2\pi}{K^2}\int_0^{\pi}\left|\sum (2l+1)\, e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)\right|^2 \sin\theta\, d\theta$$

$$= \frac{2\pi}{K^2}\sum (2l+1)^2 \sin^2\delta_l\, \frac{2}{(2l+1)}$$

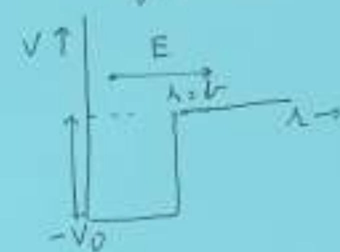
$$\boxed{\sigma = \frac{4\pi}{K^2}\sum(2l+1)\sin^2\delta_l}$$

Thus if we know phase shifts, we can calculate total cross section.

Nucleon-nucleon Scattering

The nucleon - nucleon scattering problem will be solved in centre of mass frame.

To solve nucleon - nucleon scattering problem using quantum mechanics we assume that interaction can be represented by square well potential, as we did in previous section for the deuteron. The only difference in this calculation and that of deuteron is that we are concerned with free incident particles with $E > 0$. We will simplify Schrodinger equation by assuming $l = 0$.



Solution to the square well problem for $r < b$ as well as $r > b$

$$\frac{d^2u}{dr^2} + \frac{M}{\hbar^2}\left[E - V(r) - \frac{l(l+1)\hbar^2}{Mr^2}\right]u = 0$$

but $l = 0$

$$\left.\frac{d^2u}{dr^2} + \frac{M}{\hbar^2}\left[E - V(r)\right]u = 0 \quad \right\} \quad \begin{aligned} V(r) &= -V_0 \text{ for } r<b \\ &= 0 \text{ for } r>b \end{aligned}$$

$$u''_{in} + K_2^2 u_{in} = 0 \qquad \text{for } r < b$$

$$u''_{out} + K^2 u_{out} = 0 \qquad \text{for } r > b$$

where $K_2^2 = \frac{M}{\hbar^2}(E+V_0)$ & $K^2 = \frac{ME}{\hbar^2}$

Since

$$\frac{d^2u}{dr^2} + \frac{M}{\hbar^2}\left[E - V(r)\right]u = 0$$

for $r < b$, $V = -V_0$

so $$\frac{d^2u_{in}}{dr^2} + \frac{M}{\hbar^2}\left[E + V_0\right]u_{in} = 0$$

$$\Rightarrow \frac{d^2u_{in}}{dr^2} + K_2^2 u_{in} = 0$$

$$u_{in} = A \sin K_2 r + A' \cos K_2 r$$

as $r \longrightarrow 0$, $u_{in} \longrightarrow 0$, hence $A' = 0$

$$\boxed{u_{in} = A \sin K_2 r}$$

Now for $r > b$, $V(r) = 0$

$$\frac{d^2 u_{out}}{dr^2} + \frac{ME}{\hbar^2} u_{out} = 0$$

$$\frac{d^2 u_{out}}{dr^2} + k^2 u_{out} = 0$$

$$u_{out} = B' \sin kr + B'' \cos kr$$

as

$$u_{out} = B \sin(kr + \delta_l)$$

$$u_{out} = B \sin(kr + \delta_0) \quad \text{for } l = 0 \text{ case}$$

Hence

$$u_{in} = A \sin k_2 r$$

$$u_{out} = B \sin(kr + \delta_0)$$

$$u_{in} = A \sin k_2 r$$

$$u_{out} = B \sin(kr + \delta_0)$$

Using boundary conditions on u and $\frac{du}{dr}$ at $r = b$

$$A \sin k_2 b = B \sin(kb + \delta_0)$$

$$A k_2 \cos k_2 b = B k \cos(kb + \delta_0)$$

$$\boxed{k_2 \cot k_2 b = k \cot(kb + \delta_0)}$$

Since E is known, so k and k_2 are known for a given b, we can estimate δ_0, which may be used to calculate σ_{total}.

$$k^2 = \frac{ME}{\hbar^2}, \quad k_2^2 = \frac{M}{\hbar^2}(E + V_0)$$

$$\sigma = \frac{4\pi \sin^2 \delta_0}{k^2}$$

Calculation of scattering cross section

$$\sigma = \frac{4\pi}{k^2} \sin^2\delta_0$$

since

$$k_2 \cot k_2 b = k \cot(kb + \delta_0)$$

$$\Rightarrow \quad \sin\delta_0 = \frac{\sin kb\,(k \cot kb - k_2 \cot k_2 b)}{\sqrt{k^2 + k_2^2 \cot^2 k_2 b}}$$

for $E = 10$ keV and Vo = 35 MeV.

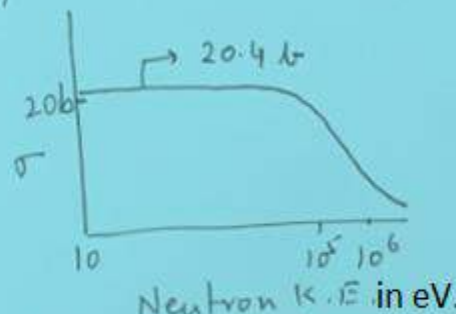
$$k_2 = \frac{\sqrt{M(V_0 + E)}}{\hbar} \simeq 0.92\ \text{fm}^{-1} \text{ for } V_0 = 35\ \text{MeV}$$

$$k = \sqrt{ME}/\hbar \simeq 0.016\ \text{fm}^{-1}$$

$$\sigma \simeq 4.6\ \text{b}$$

σ is in the range 4.6 to 5 b.

($1\ b = 10^{-28}\ m^2$)



The low energy experimental value ($\simeq 20$ b) is not in agreement with our calculated value 5 barns for s wave scattering cross setion. Solution to the discrepancy lies in the valative orientations of spins of incident and scattered nucleons.

The proton and neutron both spin $\frac{1}{2}$ particles can combine to give either 0 or 1 spin. S = 1 combination has 3 substates, while S = 0 has only one.

$$\sigma = \frac{3}{4}\sigma_t + \frac{1}{4}\sigma_s$$

for $\sigma_t = 4.6\ b$ & $\sigma = 20.4\ b$

we get $\sigma_s = 67.8\ b$

This result indicates that there is an enormous difference in singlet and triplet state σ, that is the <u>nuclear force must be spin dependent</u>.

Scattering Length

At low incident neutron energy, the cross section can be expressed in terms of the scattering length a. From the asymptotic solution of the wave equation outside the range of the nuclear force can be written as (dropping the subscript on δ_0)

$$u = r\psi = e^{i\delta} \frac{\sin(kr + \delta)}{k}$$

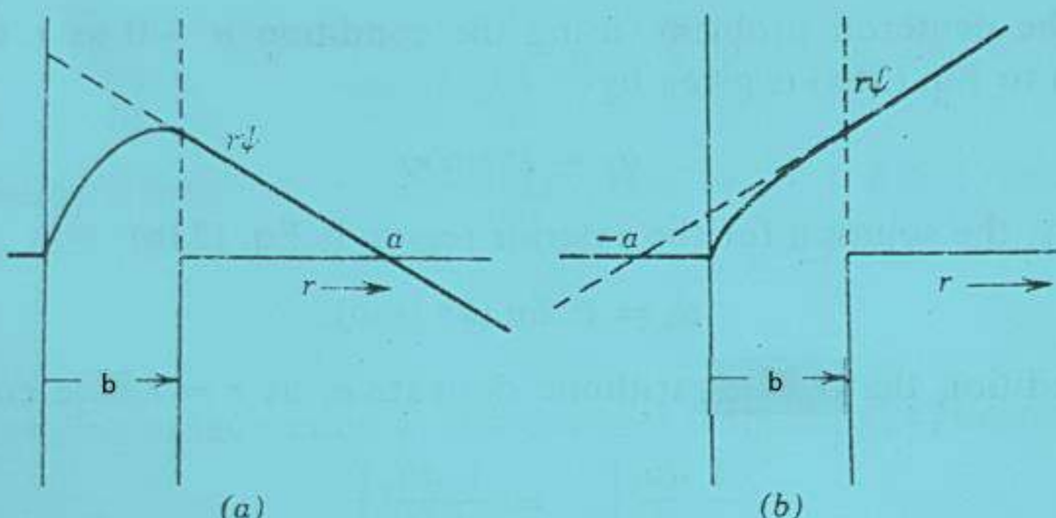
(This is unnormalized!) Clearly, for very low energy neutrons, in order that u remain finite, δ must approach zero as k does. If we define

$$\lim_{k\to 0}\left(-\frac{\sin\delta}{k}\right) = a$$

then

$$\sigma_{sc} = 4\pi\left(\frac{\sin\delta}{k}\right)^2 = 4\pi a^2$$

and a has the geometrical significance of the radius of a hard sphere from a point neutron is scattered (note: classically $\sigma_{sc} = \pi a^2$).



(a) A positive scattering length indicates that a bound state exists. (b) A negative scattering length indicates that there is no bound state of the system.

Since $\delta \to 0$ as k does, and $\delta/k = -a$, we can rewrite in the form

$$\lim_{k\to 0} u \sim \frac{kr}{k} + \frac{\delta}{k} = r - a$$

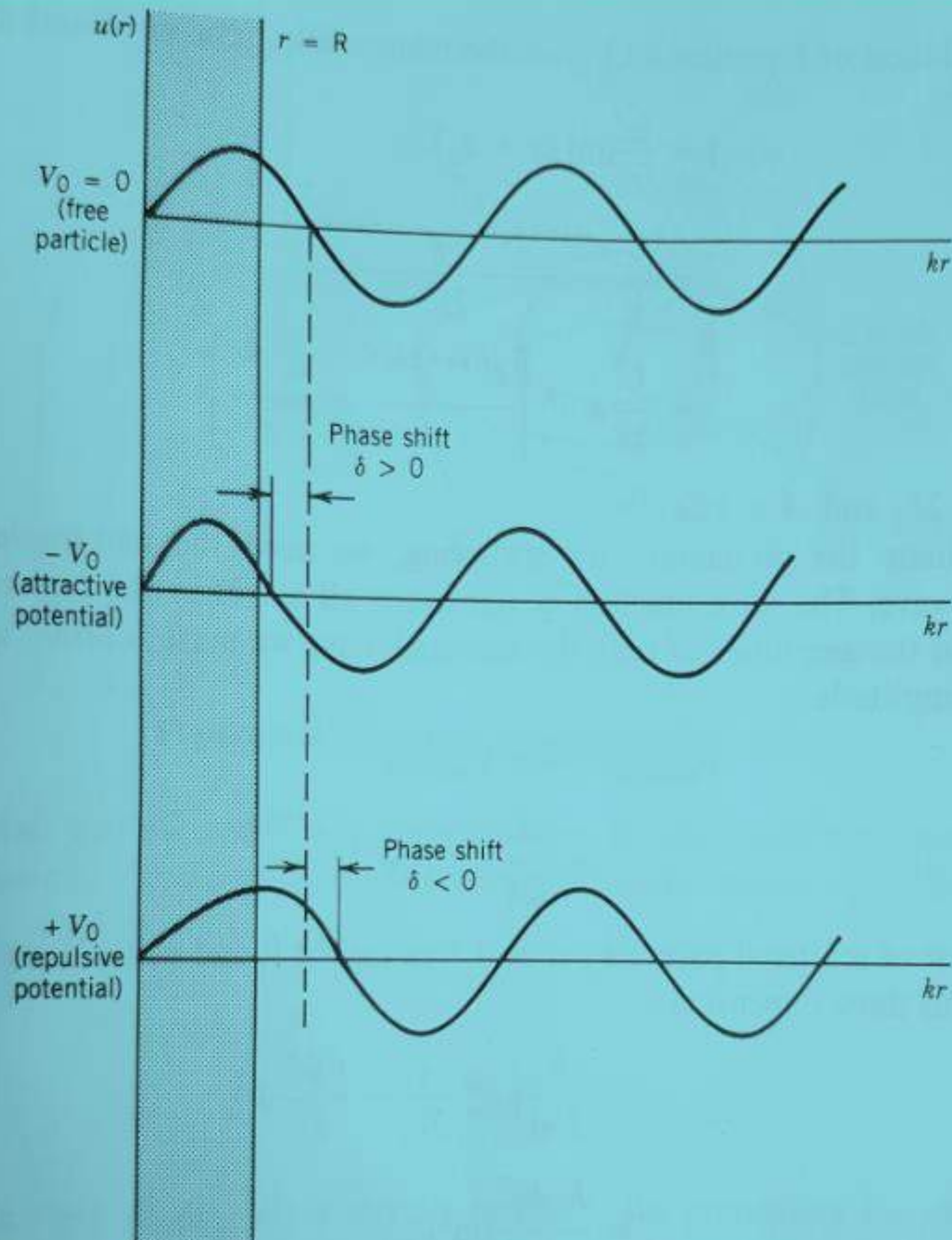
which is the equation of a straight line for $u(r)$. The scattering length,[12] a, is the intercept on the r-axis and is obtained by extrapolating the radial wave function inside the well beyond the range of force r_0. Figure illustrates the significance of the scattering length. The scattering length a is positive if the scattering state can be a bound state.

The effect of Potential $V(r)$ on the wave function

As $V(r) = V_0 \rightarrow 0$ (without scatterer being present) This is just the free particle solution as shown in the fig.

For $V(r) = -V_0$, the wave function shrinks with in the nuclear range, but for $r > b$, the wave function has the same form as the free particle, and it experiences a positive phase shift δ. The nodes of the wave function are pulled towards the origin by the attractive potential.

For $V(r) = V_0$, the repulsive potential pushes the nodes away from the origin for $r > b$ and for $r < b$, the potential stretches the wave function as depicted in the fig.



The effect of a scattering potential is to shift the phase of the scattered wave at points beyond the scattering regions, where the wave function is that of a free particle.

Criterion for Energy for s wave scattering by neutrons.

If the incident particle has velocity v, it's angular momentum relative to the target is mvb, where b is the nuclear range. The relative momentum between the nucleons must be quantized in the unit of $\hbar$; i.e.

$$mvb = l\hbar, \quad \text{where } l^2 = l(l+1)$$

For p wave scattering i.e. $l = 1$ case

$$mvb = l\hbar$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}\left[\frac{l^2\hbar^2}{mb^2}\right] = \frac{1}{2}\, l\frac{(l+1)\hbar^2c^2}{mc^2b^2}$$

here $l^2 = l(l+1)$ has been used.

$$T = \frac{1}{2} \times \frac{1 \times 2 \times 200 \times 200}{1000 \times 4}$$

where $\hbar c = 200\,\text{MeV fm}$, $mc^2 = 1000\,\text{MeV}$

$$T = \frac{1}{\cancel{2}} \times \frac{\cancel{2} \times \cancel{200} \times 200}{\cancel{1000} \times 4} = 10\ \text{MeV}$$

Hence energy requirement for p wave scattering is 10 MeV.

Hence energy of projectile less than 10 MeV in the Lab frame or 5 MeV in C.O.M frame corresponds to the 's' wave scattering.

Reference Books

- Nuclear Physics by S. N. Ghoshal
- Introductory Nuclear Physics by K. S. Krane