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Course Code- AS-201.
Mathematics -II
Laplace and Inverse Laplace
transformation

Laplace transformation:

$f(t)$: defined over positive variable t

then Laplace transformation of $f(t)$ is denoted by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = f(s)$$ where s is complex variable

$$L\{f(t)\} = f(s).$$

Laplace transf. of some special function:

$L\{1\} = \frac{1}{s}$, $s>0$

$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{\lfloor n}{s^{n+1}}$

$L\{e^{at}\} = \frac{1}{s-a}$, $s-a>0$

$L\{\cos at\} = \frac{s}{s^2+a^2}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{\sinh at\} = \frac{a}{s^2-a^2}$

$L\{\cosh at\} = \frac{s}{s^2-a^2}$

How??

$L\{1\} = \int_0^\infty e^{-st}\,1\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$

$= \frac{1}{s}$, $s>0$

$L\{t^n\} = \int_0^\infty e^{-st}\, t^n\,dt$

$st = k \Rightarrow dt = \frac{dk}{s}$

$= \int_0^\infty e^{-k} \frac{k^n}{s^n} \frac{dk}{s}$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-k} k^n\,dk$ when $t=0 \Rightarrow k=0$, $t=\infty \Rightarrow k=\infty$

$= \frac{\Gamma(n+1)}{s^{n+1}}$, $n=0,1,2\dots$

$= \frac{\lfloor n}{s^{n+1}}$.

(1) Shifting Prop^{r}:

if $L\{f(t)\} = f(s)$

$L\{e^{at} f(t)\} = f(s-a)$

(2) transf. of the Integral:

if $L\{f(t)\} = f(s)$

$L\{\int_0^t f(u)\,du\} = \frac{f(s)}{s}$

(3) $L\{f(t)\} = f(s)$

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(f(s))$

$n=1,2,3,\dots$

(4) if $L\{f(t)\} = f(s)$

then $L\{\frac{f(t)}{t}\} = \int_s^\infty f(s)\,ds.$

① Shifting Prop^r:

$$L\{f(t)\} = f(s) \Rightarrow L\{e^{at} f(t)\} = f(s-a)$$

Que-(1) $L\{e^{2t} \cdot \underline{\sin t}\} = ?$

$f(t) = \sin t$

$L\{f(t)\} = L\{\sin t\} = \frac{1}{s^2+1^2} = \frac{1}{s^2+1}$

$= f(s)$

$\Rightarrow L\{f(t)\} = f(s) = \frac{1}{s^2+1}$

Now $L\{e^{2t} \sin t\} = f(s-2)$

$= \frac{1}{(s-2)^2+1}$ Ans

(2) $L\{e^{-t} \cdot \underbrace{\cosh 2t}_{f(t)}\}$.

$f(t) = \cancel{\sinh 2t} \cosh 2t$

$L\{f(t)\} = \frac{s}{s^2-2^2} = \frac{s}{s^2-4}$

Now $L\{e^{-t} \cosh 2t\} = f(s+1)$

$= \frac{s+1}{(s+1)^2-4}$ Ans

②

L. Tranf of Integral.

If $L\{f(t)\} = f(s) \Rightarrow L\{\int_0^t f(t)\,dt\} = \frac{f(s)}{s}$

Rul Que:(1) $L\{\int_0^t \underbrace{e^{-t} \cdot \sin 2t}_{f(t)}\,dt\}$

$f(t) = e^{-t} \sin 2t$

$L\{e^{-t} \sin 2t\} = \frac{2}{(s+1)^2+4} = f(s)$

$L\{\sin 2t\} = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$

Now $L\{\int_0^t e^{-t} \sin 2t\,dt\}$

$= \frac{f(s)}{s} = \frac{\frac{2}{(s+1)^2+4}}{s}$

$= \frac{2}{s((s+1)^2+4)}$

(2) $L\{\int_0^t \underbrace{\int_0^t e^{2t} \cos t\,dt}_{f_1(t)}\,dt\}$.

$f_1(t) = \int_0^t \underbrace{e^{2t} \cos t}_{f_2(t)}\,dt$

$L\{e^{2t} \cos t\} = \frac{s-2}{(s-2)^2+1}$

$L\{\cos t\} = \frac{s}{s^2+1}$

$L\{\int_0^t e^{2t} \cos t\,dt\}$

$= \frac{f(s)}{s} = \frac{\frac{s-2}{(s-2)^2+1}}{s}$

$= \frac{s-2}{s((s-2)^2+1)}$

$L\{\int_0^t \int_0^t e^{2t} \cos t\,dt\,dt\}$

$= \frac{\frac{s-2}{s((s-2)^2+1)}}{s} = \frac{s-2}{s^2((s-2)^2+1)}$ Ans

(3) if $L\{f(t)\} = f(s) \Rightarrow L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(f(s))$

Que: $L\{e^{-t} \cdot t \cdot \sin 2t\}$.

$L\{\sin 2t\} = \frac{2}{s^2+2^2} = \frac{2}{s^2+4} = f(s)$

$L\{t \cdot \sin 2t\} = (-1)^1 \frac{d}{ds}(f(s))$

$= (-1) \frac{d}{ds}\left(\frac{2}{s^2+4}\right) = -\left(\frac{(s^2+4)\cdot 0 - 2\cdot(2s)}{(s^2+4)^2}\right.$

$= \frac{4s}{(s^2+4)^2} = f(s)$

Now $L\{e^{-t} \cdot \underline{t \cdot \sin 2t}\} = f(s+1) = \frac{4(s+1)}{((s+1)^2+4)^2}$ Ans.

(4) if $L\{f(t)\} = f(s) \Rightarrow L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s)\, ds$.

Que $L\left\{\frac{\cos at - \cos bt}{t}\right\}$.

Here $f(t) = \cos at - \cos bt$

$L\{f(t)\} = L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\, ds$

$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{1+\frac{a^2}{s^2}}{1+b^2/s^2}\right)\right]_s^\infty$

$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$

$= \log\left[\left(\frac{s^2+b^2}{s^2+a^2}\right)\right]^{1/2}$ Ans

Inverse laplace transformation:

(1) $L^{-1}\left\{\frac{1}{(s+1)(s-1)}\right\}$ => by Partial Fraction

(2) $L^{-1}\left\{\frac{s^2+2s+3}{s^3}\right\}$

(3) $L^{-1}\left\{\frac{2}{s(s^2+4)}\right\}$

$L^{-1}\left\{\frac{1}{(s+1)(s-1)}\right\} = L^{-1}\left\{\frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)\right\}$

$= \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\}$

$= \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$

$= \frac{1}{2}(e^{t} - e^{-t})$

Ans

$L^{-1}\left\{\frac{s^2+2s+3}{s^3}\right\}$

$L^{-1}\left\{\frac{s^2}{s^3} + \frac{2s}{s^3} + \frac{3}{s^3}\right\}$

$L^{-1}\left\{\frac{1}{s} + \frac{2}{s^2} + \frac{3}{s^3}\right\}$

$= L^{-1}\left\{\frac{1}{s}\right\} + 2L^{-1}\left\{\frac{1}{s^2}\right\} + 3L^{-1}\left\{\frac{1}{s^3}\right\}$

$= 1 + 2\cdot\frac{t}{1!} + 3\cdot\frac{t^2}{2!}$

$= 1 + 2t + \frac{3}{2}t^2$ Ans.

if $L\{f(t)\} = f(s)$

$\Rightarrow L\left\{\int_0^t f(u)\,du\right\} = \frac{f(s)}{s}$

$\Rightarrow L^{-1}\{f(s)\} = f(t)$

$\Rightarrow L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t f(u)\,du$

Qus $L^{-1}\left\{\frac{2}{s(s^2+4)}\right\}$.

$L^{-1}\left\{\frac{\left(\frac{2}{s^2+4}\right)}{s}\right\}$

$\Rightarrow f(s) = \frac{2}{s^2+4}$.

$f(t) = \sin 2t$

$\Rightarrow L^{-1}\left\{\frac{\frac{2}{s^2+4}}{s}\right\} = \int_0^t f(u)\,du$

$= \int_0^t \sin 2u\,du$

$= \left[-\frac{\cos 2u}{u}\right]_0^t$

$=$ Ans.

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Imp:

Convolution Theorem for Inverse Laplace Transformation

if $L^{-1}\{f(s)\} = f(t)$ $\quad L^{-1}\{g(s)\} = g(t)$

then $\boxed{L^{-1}\{f(s)\cdot g(s)\} = F * G = \int_0^t f(u)\, g(t-u)\, du}$

Exercise: $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}. = L^{-1}\left\{\underbrace{\frac{s}{s^2+a^2}}_{f(s)} \cdot \underbrace{\frac{s}{s^2+b^2}}_{g(s)}\right\}.$

$\Rightarrow f(t) = \cos at \quad g(t) = \cos bt$

$\Rightarrow L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos au \cdot \cos b(t-u)\, du$

as $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$\Rightarrow L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \frac{1}{2}\left(\cos((a-b)u+bt) + \cos((a+b)u - bt)\right) du$

$= \frac{1}{2}\left[\frac{\sin(a-b)u+bt}{a-b} + \frac{\sin(a+b)u-bt}{a+b}\right]_0^t$

$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right]$

$= \frac{a\sin at - b\sin bt}{a^2-b^2}$. Ans

(2): $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\underbrace{\frac{s}{s^2+a^2}}_{f(s)} \cdot \underbrace{\frac{s}{s^2+a^2}}_{g(s)}\right\}$

(3) $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = L^{-1}\left\{\underbrace{\frac{1}{(s+a)}}_{f(s)} \cdot \underbrace{\frac{1}{s+b}}_{g(s)}\right\}$

(4) $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} \rightarrow L^{-1}\left\{\underbrace{\frac{1}{s}}_{f(s)} \cdot \underbrace{\frac{1}{s^2+4}}_{g(s)}\right\}$.

(5) $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$.

$\rightarrow L^{-1}\left\{\frac{1}{s^2} \cdot \frac{1}{(s+1)^2}\right\}$

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Transformation of Derivatives:

$$L\{f'(t)\} = \int_0^{\infty} e^{-st} f'(t)\,dt = \left[e^{-st} \cdot f(t)\right]_0^{\infty} - \int_0^{\infty} -se^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\int_0^{\infty} e^{-st} f(t)\,dt$$

$$= -f(0) + s \cdot L\{f(t)\}$$

$$= s \cdot L\{f(t)\} - f(0)$$

$$\boxed{L\{f'(t)\} = s\,f(s) - f(0)} \quad \text{--- (1)}$$

Now $L\{f''(t)\} = \int_0^{\infty} e^{-st} \cdot f''(t)\,dt = \left[e^{-st} f'(t)\right]_0^{\infty} - \int_0^{\infty} -se^{-st} \cdot f'(t)\,dt$

$$= -f'(0) + s\int_0^{\infty} e^{-st} f'(t)\,dt$$

$$= -f'(0) + s\left(sf(s) - f(0)\right) \quad \text{(by equ (1))}$$

$$\boxed{L\{f''(t)\} = s^2 f(s) - sf(0) - f'(0)}$$

Similarily $L\{f'''(t)\} = s^3 f(s) - s^2 f(0) - sf'(0) - s^0 f''(0)$

Rule. $\begin{cases}\text{decreasing power of } s \\ \text{increasing order of derivative of } f(0).\end{cases}$

$$L\{f''''(t)\} = s^4 f(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - s^0 f'''(0)$$

$$L\{f'''''(t)\} = s^5 f(s) - s^4 f(0) - s^3 f'(0) - s^2 f''(0) - sf'''(0) - s^0 f''''(0)$$

Transformation of Derivatives:

if $L\{f(t)\} = f(s) \Rightarrow L\{f'(t)\} = sf(s) - f(0)$

(ii) $L\{f''(t)\} = s^2 f(s) - sf(0) - f'(0)$

$L\{f'''(t)\} = s^3 f(s) - s^2 f(0) - sf'(0) - f''(0)$.

Que: $y''' + 2y'' - y' - 2y = 0$ given $y(0) = y'(0) = 0$
$y''(0) = 6$.

Let $Y(s) = \bar{y} = L\{y\}$ and operating L operator on both sides

$L\{y'''\} + 2L\{y''\} - L\{y'\} - 2L\{y\} = L\{0\}$

$(s^3\bar{y} - s^2 y(0) - s y'(0) - y''(0)) + 2(s^2\bar{y} - s y(0) - y'(0))$
$- (s\bar{y} - y(0)) - 2\bar{y} = 0$

$\Rightarrow$ Now after putting the value of Initial Cond^n

we get $(s^3 + 2s^2 - s - 2)\bar{y} = 6$

$\Rightarrow \bar{y} = \dfrac{6}{s^3+2s^2-s-2} = \dfrac{6}{(s-1)(s+1)(s+2)}$

$\bar{y} = \dfrac{1}{s-1} - \dfrac{3}{s+1} + 2\cdot\dfrac{1}{s+2}$ (by Partial Fraction).

$\Rightarrow \bar{y} = \cancel{L\{y}$

$\bar{y} = L\{y\} = \dfrac{1}{s-1} - \dfrac{3}{s+1} + \dfrac{2}{s+2}$

$\Rightarrow y = L^{-1}\left\{\dfrac{1}{s-1}\right\} - 3L^{-1}\left\{\dfrac{1}{s+1}\right\} + 2L^{-1}\left\{\dfrac{1}{s+2}\right\}$

$= e^{t} - 3e^{-t} + 2\cdot e^{-2t}$.

$\Rightarrow \boxed{y = e^{t} + 2e^{-2t} - 3e^{-t}}$ Ans.

Simultaneous L. diff. Equation with const. Coeff.

que: $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$, $x = y = 0$ when $t=0$

let $L\{\frac{dx}{dt}\} - L\{x'\} = \bar{x}$ & $L\{\frac{dy}{dt}\} = L\{y'\} = \bar{y}$

Now operating L. Transf. on both equation, we get

$L\{x'\} + 5L\{x\} - 2L\{y\} = L\{t\}$ — (I)

& $L\{y'\} + 2L\{x\} + L\{y\} = L\{0\}$ — (II)

equ (1) $\Rightarrow$ $s\bar{x} - x(0) + 5\bar{x} - 2\bar{y} = \frac{1}{s^2}$

(2) $\Rightarrow$ $s\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0$

or $\left.\begin{array}{l}(s+5)\bar{x} - 2\bar{y} = \frac{1}{s^2} \\ 2\bar{x} + (s+1)\bar{y} = 0\end{array}\right\} \Rightarrow$ $\bar{x} = \frac{s+1}{s^2(s+3)^2}$

$\bar{y} = -\frac{2}{s^2(s+3)^2}$

$$\bar{x} = \frac{s+1}{s^2(s+3)^2}$$

$$L\{x\} = \frac{s+1}{s^2(s+3)^2}$$

operating L^{-1} on both side

$\Rightarrow$ $x = L^{-1}\left\{\frac{s+1}{s^2(s+3)^2}\right\}$ Partial Fraction

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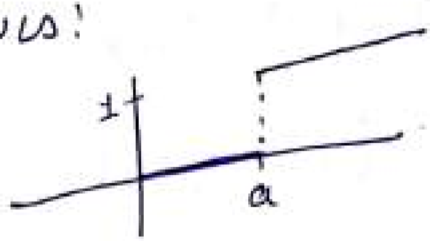
Unit step function:

(1) Unit step func' is defined as follows:



$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a e^{-st} u(t-a)\,dt + \int_a^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st} \cdot 1\,dt = \left[\frac{-e^{-st}}{s}\right]_a^\infty$$

$$= \frac{e^{-as}}{s}$$

$$\Rightarrow \boxed{L\{u(t-a)\} = e^{-as}/s}$$

(2) Unit Impulse function: $L\{\delta(t-a)\} = e^{-as}$.

(3) Periodic func^c: $L\{f(t)\} = \dfrac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$

where $f(t)$ is P. func^c with period T i.e. $f(t+T) = f(t)$

(4) Bessel function: $L\{J_0(x)\} = \dfrac{1}{\sqrt{s^2+1}}$

(5) Error function: $L\{\text{erf}(\sqrt{x})\} = \dfrac{1}{s\sqrt{s+1}}$

where $\text{erf}(\sqrt{x}) = \dfrac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-t^2}\,dt$.

Exercise:

(1) $L\{\sin 4t\} = \frac{4}{s^2+4^2} = \frac{4}{s^2+16}$

(2) $L\{e^{2t}.\sin 2t\}$ = as $L\{\sin 2t\} = \frac{2}{s^2+4} \Rightarrow L\{e^{2t}.\sin 2t\} = \frac{2}{(s-2)^2+4}$

(3) $L\{e^{-t}.\cosh 3t\}$, as $L\{\cosh 3t\} = \frac{s}{s^2-3^2}$

$\Rightarrow L\{e^{-t}.\cosh 3t\} = \frac{s+1}{(s+1)^2-9}$

(4) $L\{\int_0^t \underbrace{e^{-t}\sin t}_{f(t)}\, dt\}$

$L\{e^{-t}.\sin t\} = \frac{1}{(s+1)^2+1} = f(s)$

$L\{\sin t\} = \frac{1}{s^2+1}$

$L\{\int_0^t e^{-t}\sin t\, dt\} = \frac{f(s)}{s} = \frac{\left(\frac{1}{(s+1)^2+1}\right)}{s}$

$= \frac{1}{s.((s+1)^2+1)}$ Ans

(5) $L\{\int_0^t\int_0^t \underbrace{e^{2t}.\cos t}\, dt\, dt\}$

$L\{e^{2t}.\cos t\} = \frac{s-2}{(s-2)^2+1}$

$L\{\cos t\} = \frac{s}{s^2+1}$

Now $L\{\int_0^t e^{2t}\cos t\, dt\}$

$= \frac{f(s)}{s} = \frac{\left(\frac{s-2}{(s-2)^2+1}\right)}{s}$

$\Rightarrow L\{\int_0^t\int_0^t e^{2t}\cos t\, dt\, dt\} = \left(\frac{s-2}{s^2((s-2)^2+1)}\right)$

(6) $L\{e^{2t}.t.\cos t\}$

$L\{\cos t\} = \frac{s}{s^2+1}$

$L\{t.\cos t\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+1}\right)$

$= -\left(\frac{(s^2+1).1 - s(2s)}{(s^2+1)^2}\right)$

$= \frac{s^2-1}{(s^2+1)^2} = f(s)$

$L\{e^{2t}.t.\cos t\} = \frac{(s-2)^2-1}{((s-2)^2+1)^2}$ Ans

(7) $L\{\frac{\cos at - \cos bt}{t}\}$.

$= L\{\cos at - \cos bt\}$

$\Rightarrow = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = f(s)$

Now $L\{\frac{\cos at - \cos bt}{t}\}$

$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$

$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$

or $= \frac{1}{2}\left[\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]_s^\infty$

$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$

$= \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$ Ans