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Course - B.Tech - IV Sem, CSE
Topic - Coding Theory. - AS-404.

* Alphabet : $\{0, 1\}$ Binary bit.
* message and Code word both Can be represent by $\{0, 1\}$ only.
* n : tuple of Code word
* m : " " message , $n > m$.

$$\boxed{\text{message}} \xrightarrow[E]{\text{Encoding Rule}} \boxed{\text{Code word}} \xrightarrow[D]{\text{decoding Rule}} \boxed{\text{message}}$$

* $E : B^m \longrightarrow B^n$, $|B^m| = 2^m$
* $D : R \longrightarrow B^m$, $R \subseteq B^n$
* if $m = 3$, $\Rightarrow$ $|B^3| = 2^3 = 8$
  $\Rightarrow$ set of message $= \{000, 100, 010, 001, 110, 101, 011, 111\}$
* if $n = 4 \Rightarrow$ $B^4 = \{$ Collection of Code words of 4 tuples $\}$.

Example: suppose $E(101) = 101101$ } Error detecting case.
$E(100) = 101101$.

If we know the Encoding Rule, suppose it is

$E(a_1 a_2 a_3) = a_1 a_2 a_3 \, a_1 a_2 a_3$

$\Rightarrow E(100) = 100100$

$E(101) = 101101$

This is the case of Error Detecting and also error correcting case.

Hamming Distance & Hamming Weight:

Hamming Weight: The number of non zero entry in any code is called Hamming Weight of the code.

Exa $x_1 = 110110 \Rightarrow$ H. weight of $x_1 = 4$.

$x_2 = 10101 \Rightarrow$ H. weight of $x_2 = 3$

Hamming distance: Number of position between two code x & y, where the code word have diff. symbols.

Example: If $x = 101011$ then $d(x,y) = 2$ $d(z,x) = 4$.

$y = 101101$ $d(y,z) = 2$

$z = 110101$

Theorem Result: (where $K \geq 0$.)

(i) we can detect K or fewer error iff.

min H-distance $\geq K+1$.

(ii) we can correct K or fewer error iff.

min H-distance $\geq 2K+1$.

Example: (6) Given that $E: B^2 \to B^6$ defined by

$E(00) = 000000 = x_1$

$E(01) = 010101 = x_2$

$E(10) = 101010 = x_3$

$E(11) = 111111 = x_4$

Find the number of error which can be detected and corrected?

$d(x_2, x_3) = 6$

$d(x_2, x_4) = 3$

$d(x_3, x_4) = 3$

$\Rightarrow$ min H. distance $= 3$

$d(x_1 x_2) = 3$

$d(x_1 x_3) = 3$

$d(x_1 x_4) = 6$

$3 \geq 2K+1$

$\Rightarrow 2K \leq 3-1$

$2K \leq 2$

$K \leq 1$

$\Rightarrow$ we can correct 1 error only.

for Error detecting case $3 \geq K+1$

$\Rightarrow K \leq 3-1$

$K \leq 2$.

$\Rightarrow$ we can detect 2 or fewer error.

Generator Matrix: If $E: B^m \to B^n$ be a Encoding function then the matrix of the form

$$\boxed{G = \left[I_{m\times m} : A_{m\times(n-m)} \right]_{m\times n}}$$

is called generator matrix

where

$I_{m\times m}$: Identity matrix

* Relation b/w message, code word and G. matrix

$$\boxed{b = a \cdot G}$$

row matrix

b: Codeword Generated by a

a: message.

G: Generator Matrix.

Parity check matrix:- It is denoted by H.

$$H = \left[A^T : I_{(n-m)\times(n-m)} \right]_{(n-m)\times n}$$

Note: (i) ~~If~~ The Code word b is valid or we can say correctly transmitted if

$$\boxed{b \cdot H^T = 0 \quad \text{or} \quad H b^T = 0}$$

if $bH^T \neq 0 \Rightarrow$ code are not correctly transmitted

(ii) Here matrix multiplication is defined over adding modulo 2 i.e '$+_2$'.

Que : Given that $G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$ find

(I) All Codewords

(II) Min H. weight & Min H. Distance.

(III) How many error we can detect and correct.

(IV) Parity check Matrix.

(V) Check the following code's are correctly transmitted or not $a = 01000$ $b = 11010$

Soln: $G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}_{2\times 5}$ & $G = [I_{m\times m} : A_{m\times(n-m)}]_{m\times n}$

$\Rightarrow m = 2,\ n = 5,\ n-m = 5-2 = 3$

$\Rightarrow G = [I_{2\times 2} : A_{2\times(3)}]_{2\times 5}$

$\Rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

if $m = 2$, $n = 5$ $\Rightarrow E : B^2 \to B^5$

$\Rightarrow |B^2| = 2^2 = 4 = 2^m$

$\Rightarrow$ set of message $B^2 = \{00, 10, 01, 11\}$.

$B = a\,G$ b : code word, a : message, G : Generator matrix

Now

$\Rightarrow$ $b_1 = (00)\,G = (00) \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$

$= [0\ 0\ 0\ 0\ 0] = x_1$

$b_2 = (10)\,G = (10) \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$

$= [1\ 0\ 1\ 1\ 1] = x_2$

$b_3 = (01)\,G = [0\ 1\ 1\ 0\ 1] = x_3$

$b_4 = (11)\,G = [1\ 1\ 0\ 1\ 0] = x_4$

$d(x_1 x_2) = 4$
$d(x_1 x_3) = 3$
$d(x_1 x_4) = 3$
$d(x_2 x_3) = 3$
$d(x_2 x_4) = 3$
$d(x_3 x_4) = 4.$

min H. distance = 3.
How many error we can detect!

$3 \geq K+1 \Rightarrow K \leq 3-1$
$\Rightarrow K \leq 2$

$\Rightarrow$ we can detect 2 or fewer error Here.

Similarily How many error we can correct Ha
$3 \geq 2K+1 \Rightarrow 2K \leq 3-1$
$\Rightarrow 2K \leq 3-1$
$\Rightarrow K \leq 1$
$\Rightarrow$ we can correct 1 error Here.

$$H = [A^T : I_{(n-m)\times(n-m)}]_{(n-m)\times n}.$$

$$\Rightarrow H = [A^T : I_{(5-2)\times(5-2)}]_{(5-2)\times 5}$$

$$= [A^T : I_{3\times 3}]_{3\times 5}$$

$$\boxed{H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}_{3\times 5}}$$

$H \cdot b^T = 0 \Rightarrow$ code b is correctly transmitted

if $Hb^T \neq 0 \Rightarrow$ code is not valid.

$a = 01000$ $\qquad b = 11010$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{1\cdot 0 + 1\cdot 1 + 1\cdot 0 + 0\cdot 0 + 0\cdot 0} \\ +2 \\ \underline{1\cdot 0 + 0\cdot 1 + 0\cdot 0 + 0\cdot 0 + 0\cdot 0} \\ +2 \\ \underline{1\cdot 0 + 1\cdot 1 + 0\cdot 0 + 0\cdot 0 + 1\cdot 0} \\ +2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

= code a is not correctly transmitted. Ans.
b " " " "