GAME THEORY-1

Jai Lakshmi Sharma

GAME THEORY

Basic objective:

- To introduce game theory and some of its basic terminologies
- To introduce Zero-sum game
- To study game with pure strategy
- To study game with mixed strategy

Game theory is the study of mathematical models of strategic interaction among rational decision-makers. It has applications in all fields of social science, as well as in logic, science and computer science. Originally, it addressed zero-sum games, in which each participant's gains or losses are exactly balanced by those of the other participants. Today, game theory applies to a wide range of behavioural relations, and is now an umbrella term for the science of logical decision making in humans, animals, and computers.

Game theory is the formal study of conflict, competition and cooperation. Game theoretic concepts apply whenever the actions of several agents are interdependent. These agents may be individuals, groups, firms, or any combination of these. The concepts of game theory provide a language to formulate, structure, analyze, and understand strategic scenarios. As a mathematical tool for the decision-maker the strength of game theory is the methodology it provides for structuring and analyzing problems of strategic choice. The process of formally modelling a situation as a game requires the decision-maker to enumerate explicitly the players and their strategic options, and to consider their preferences and reactions. The discipline involved in constructing such a model already has the potential of providing the decision-maker with a clearer and broader view of the situation. This is a “prescriptive” application of game theory, with the goal of improved strategic decision making.

History

- The earliest example of a formal game-theoretic analysis is the study of a duopoly by Antoine Cournot in 1838. The mathematician Emile Borel suggested a formal theory of games in 1921, which was furthered by the mathematician John von Neumann in 1928 in a “theory of parlor games.” Game theory was established as a field in its own right after the 1944 publication of the monumental volume Theory of Games and Economic Behaviour by von Neumann and the economist Oskar Morgenstern.
- In 1950, John Nash demonstrated that finite games have always have an equilibrium point, at which all players choose actions which are best for them given their opponents’ choices. This central concept of noncooperative game theory has been a focal point of analysis since then. In the 1950s and 1960s, game theory was broadened theoretically and applied to problems of war and politics. Since the 1970s, it has driven a revolution in economic theory. Additionally, it has found applications in sociology and psychology, and established links with evolution and biology. Game theory received special attention in 1994 with the awarding of the Nobel prize in economics to Nash, John Harsanyi, and Reinhard Selten.

- At the end of the 1990s, a high-profile application of game theory has been the design of auctions. Prominent game theorists have been involved in the design of auctions for allocating rights to the use of bands of the electromagnetic spectrum to the mobile telecommunications industry. Most of these auctions were designed with the goal of allocating these resources more efficiently than traditional governmental practices, and additionally raised billions of dollars in the United States and Europe.

Terminologies:

- **Game:** game is a formal model of an interactive situation. It typically involves several players; a game with only one player is usually called a decision problem. The formal definition lays out the players, their preferences, their information, the strategic actions available to them, and how these influences the outcome.
- **Game theory:** Game theory is the study of the ways in which *interacting choices* of *economic agents* produce *outcomes* with respect to the *preferences* (or *utilities*) of those agents, where the outcomes in question might have been intended by none of the agents. Their preferences are known here as strategies which are given or defined by rules. For each combination of players and possible strategies, there is a pay-off.

- **Player:** Each participant in the game is called a player . A central assumption in many variants of game theory is that the players are rational. A rational player is one who always chooses an action which gives the outcome he most prefers, given what he expects his opponents to do. An economically rational player is one who can

(i) assess outcomes, in the sense of rank-ordering them with respect to their contributions to her welfare;

(ii) calculate paths to outcomes, in the sense of recognizing which sequences of actions are probabilistically associated with which outcomes;

(iii) select actions from sets of alternatives (which we'll describe as 'choosing' actions) that yield her most-preferred outcomes, given the actions of the other players.

- **Strategy:** The strategy for the player is the list of all possible actions(moves, decision alternatives or courses of action).
- **Optimal Strategy:** The particular strategy that optimizes a player's gains or losses , without knowing the competitor's strategies, is called Optimal strategy.
- **Value of the Game:** The expected outcome, when players use their optimal strategy is called value of the game.
- **Fair game:** The game whose value v = 0 is known as zero sum game or fair game.
- **Pay-off matrix:** A pay - off matrix is a table, which shows how payments should be made at end of a play or the game.

Generally, there are two types of strategies followed by players in a game, Pure strategy and mixed strategy.

- **Pure strategy:** pure strategy is a decision rule always to select the same course of action.
- **Mixed strategy:** A mixed strategy is that in which a player decides, in advance to choose on of his course of action in accordance with some fixed probability distribution.
- **Two-person zero-sum game:** A game where two persons are playing the game and the sum of gains and losses is equal to zero, the game is known as Two-Person Zero-Sum Game (TPZSG).

Two-person zero-sum game-Assumptions

- Each player has finite number of possible strategies to him . number of strategies may or may not be the same for each player.
- List of strategies of each player and the amount of gain or loss on an individual's choice of strategy is known toeach player in advance.
- Players are rational.
- One player tries to maximize gain other wants to minimize loss.
- Both players select and announce their strategies simultaneously.
- The pay-off is fixed and known in advance.

Two-person zero-sum game-Pure strategy

If two players in the game are A and B and it is zero sum game, only one players' pay-off is required to evaluate the decision. By convention, the pay-off table for the player whose strategies are represented by rows(say player A)is constructed.

- **Maximin principle :** For player A the minimum value in each row represents the least gain to him. After choosing minimum from each row he will choose maximum out of them,as he tries to maximize his gain.
- **Minimax principle: F**or player B the maximum value in each column represents the maximum loss to him. After choosing maximum from each column he will choose minimum out of them , as he tries to minimize his loss.

Let us take one example

Solve the game:

	B1	**B2**	**B3**
A1	12	-8	-2
A2	6	7	3
A3	-10	-6	-2

In this game Player A have three strategies, A_1, A_2 and A_3. and player B have three strategies B_1, B_2 and B_3 .

	B1	B2	B3	Row minimum
A1	12	-8	-2	-8
A2	6	7	3	3
A3	-10	-6	-2	-10
Column Maximum	12	7	3	

player A:He will first all see minimum gain from all his strategies, then will choose maximum out of that i.e. Maximin(maximum out of minimum)

player B:He will first all see maximum loss from all his strategies, then will choose minimum out of that i.e. Minimax(minimum out of maximum).

IF Maximin value=minimax value

then game has a saddle(equilibrium) point.

In above game, Maximin=minimax=3

- Optimal strategy for player A is A_2.
- Optimal strategy for player B is B_3.
- Value of the game is 3.
- Game is not fair as value of game is not 0.

Two-person zero-sum game-Mixed strategy

When a game doesn't has a saddle point

i.e. Maximin value$\neq$minimax value then we use mixed strategy to solve the game.

In case there is no saddle point the given game matrix ($m \times n$) may be reduced to $m \times 2$ or $2 \times n$ or 2×2 matrix, which will help us to proceed further to solve the game. The ultimate way is we have to reduce the given matrix to 2×2 to solve mathematically.

So Game reduced to $m \times 2$ or $2 \times n$ or 2×2 matrix by the principle of dominance can be solved with

1.Algebric method

2. Graphic method

Rules for dominance

The general rules of dominance can be formulated as below

1. If all the elements of a column (say i^{th} column) are greater than or equal to the corresponding elements of any other column (say j^{th} column), then i^{th} column is dominated by j^{th} column.

2. If all the elements of r^{th} row are less than or equal to the corresponding elements of any other row, say s^{th} row, then r^{th} row is dominated by s^{th} row.

3. A pure strategy of a player may also be dominated if it is inferior to some convex combinations of two or more pure strategies, as a particular case, inferior to the averages of two or more pure strategies.

Algebraic Method

- **If** two players in the game are A and B . A have two strategies A_1 and A_2 and B have B_1 and B_2 i.e. the game is 2 ×2 . Since the game cannot be solved with pure strategies i.e. it doesn't have saddle point . So the game can be solved with mixed strategy. If A's probability of playing A_1 strategy is p_1 and A_2 is p_2 where $p_2=(1-p_1)$ and B's probability of playing B_1 strategy is q_1 and B_2 is q_2 where $q_2= (1-q_1)$.

		q_1	q_2
		B_1	B_2
p_1	A_1	a_{11}	a_{12}
p_2	A2	a_{21}	a_{22}

- $p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$

- $p_2 = 1 - p_1$

- $q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$

- $q_2 = 1 - q_1$
- Value of the game= $\frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$

Take one example,

Solve the game:

	B1	**B2**	**B3**
A1	9	8	-7
A2	3	-6	4
A3	6	7	7

	B1	B2	B3	Row minimum
A1	9	8	-7	-7
A2	3	-6	4	-6
A3	6	7	7	6
Column maximum	9	8	7	

Here Maximin value= 6

minimax value =7

i.e. Maximin value$\neq$minimax value

So the game has no saddle point .This game cannot be solved with pure strategy.

For solving this game with mixed strategy it has to be reduced to 2×2 game. For that we need to apply principle of dominance.

- 1. **Rule of rows**: As all the elements of A_2 is less or equal to A_3.so row A_2 is dominated by A_3 so A_2 can be eliminated.

	B1	**B2**	**B3**
A1	9	8	-7
A3	6	7	7

- **Rule of column** : In above table, all the element of column B_2 is more or equal to that of B_3. so B_2 is dominated by B_3 and can be eliminated.

	B1	**B3**
A1	9	-7
A3	6	7

- Let we assume that A’s probability of playing A_1 strategy is p_1 and A_3 is p_2 where $p_2=(1-p_1)$.
- Let we assume that B’s probability of playing B_1 strategy is q_1 and B_3 is q_2 where $q_2=(1-q_1)$.

- $p_1=\frac{a_{22}-a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})}=\frac{7-6}{9+7-(6-7)}=\frac{1}{17}$

  $p_2=1-p_1=1-\frac{1}{17}=\frac{16}{17}$

- $q_1=\frac{a_{22}-a_{12}}{a_{11}+a_{22}-(a_{12}+a_{21})}=\frac{7-(-7)}{9+7-(6-7)}=\frac{14}{17}$

  $q_2=1-q_1=1-\frac{14}{17}=\frac{3}{17}$

Expected value of the game =V=$\frac{a_{22}a_{11}-a_{12}a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})}$= $\frac{9\times7-6\times(-7)}{9+7-(6\ -7)}$=$\frac{105}{17}$

optimal strategy for player A will be $(\frac{1}{17}, 0, \frac{16}{17})$

optimal strategy for player A will be $(\frac{14}{17}, 0, \frac{3}{17})$

- Graphical method for solving $m \times 2$ or $2 \times n$ games will discussed in next e-content.