**Unit - II**

**Linear Systems-** Let us consider a system of first order differential equations of the form

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y) \qquad (1)$$

Where t is an independent variable. And x & y are dependent variables.

The system (1) is called a linear system if both F(x, y) and G(x, y) are linear in x and y.

Also system (1) can be written as
$$\frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t)$$
$$\frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t) \qquad (2)$$

Where $a_i(t)$, $b_i(t)$ and $f_i(t)$ $\forall i = 1, 2$ are continuous functions on $[a, b]$.

**Homogeneous and Non-Homogeneous Linear Systems-** The system (2) is called a homogeneous linear system, if both $f_1(t)$ and $f_2(t)$ are identically zero and if both $f_1(t)$ and $f_2(t)$ are not equal to zero, then the system (2) is called a non-homogeneous linear system.

**Solution-** A pair of functions $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ defined on $[a, b]$ is said to be a solution of (2) if it satisfies (2).

Example-
$$\frac{dx}{dt} = 4x - y \quad ........A$$
$$\frac{dy}{dt} = 2x + y \quad ........B \qquad (3)$$

From A, $y = 4x - \frac{dx}{dt}$ putting in B we obtain $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$ is a 2nd order differential equation. The auxiliary equation is $m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$ so $\begin{matrix} x = e^{2t} \\ x = e^{3t} \end{matrix}$ putting $x = e^{2t}$ in A, we obtain $y = 2e^{2t}$ again putting $x = e^{3t}$ in A, we obtain $y = e^{3t}$. Therefore the solutions of (3) are

$$\begin{matrix} x = e^{2t} \\ y = 2e^{2t} \end{matrix} \text{ and } \begin{matrix} x = e^{3t} \\ y = e^{3t} \end{matrix} \qquad (4)$$

**Theorem**-1 If $t_0$ is any point of $[a,b]$ and $x_0$ & $y_0$ are any two numbers, then the system (2) has a unique solution $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ with $\begin{matrix} x(t_0) = x_0 \\ y(t_0) = y_0 \end{matrix}$.

**Theorem-2** If the homogeneous system
$$\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned} \qquad (5)$$

has two solutions $\begin{matrix} x = x_1(t) \\ y = y_1(t) \end{matrix}$ and $\begin{matrix} x = x_2(t) \\ y = y_2(t) \end{matrix}$ (6)

on $[a,b]$. Then $\begin{matrix} x = c_1x_1(t) + c_2x_2(t) \\ y = c_1y_1(t) + c_2y_2(t) \end{matrix}$ (7)

is also a solution of (5) on $[a,b]$ for any two constants $c_1$ and $c_2$.

**Theorem-3** If the two solutions $\begin{matrix} x = x_1(t) \\ y = y_1(t) \end{matrix}$ and $\begin{matrix} x = x_2(t) \\ y = y_2(t) \end{matrix}$ (6) of the homogeneous system (5) have a wronskian $W(t)$ that does not vanish on $[a,b]$, then $\begin{matrix} x = c_1x_1(t) + c_2x_2(t) \\ y = c_1y_1(t) + c_2y_2(t) \end{matrix}$ (7) is a general solution of homogeneous system (5) on $[a,b]$.

**Note-** The wronskian W(t) of the solutions (4) is

$$W(t) = \begin{vmatrix} e^{3t} & e^{2t} \\ e^{3t} & 2e^{2t} \end{vmatrix}$$
$$= e^{5t}$$

**Theorem -4** The wronskian W(t) of two solutions (6) of homogeneous system (5) is either identically zero or nowhere zero on $[a,b]$ i.e

$W(t) = 0$ (linearly dependent) or $W(t) \neq 0$ (linearly independent).

The wronskian W(t) satisfies the differential equation, $\frac{dW}{dt} = [a_1(t) + b_2(t)]W$ and on integrating between the limits 0 to t we obtain

$$W(t) = ce^{\int_0^t [a_1(t)+b_2(t)]dt}.$$

**Theorem -5** If the two solutions $\begin{matrix} x = x_1(t) \\ y = y_1(t) \end{matrix}$ and $\begin{matrix} x = x_2(t) \\ y = y_2(t) \end{matrix}$ of homogeneous system (5) are linearly independent on $[a, b]$ and if $\begin{matrix} x = x_p(t) \\ y = y_p(t) \end{matrix}$ is any particular solution of non-homogeneous system (2) on $[a, b]$, then $\begin{matrix} x = c_1 x_1(t) + c_2 x_2(t) + x_p(t) \\ y = c_1 y_1(t) + c_2 y_2(t) + y_p(t) \end{matrix}$ is a general solution of of non-homogeneous system (2) on $[a, b]$.

**Example**- Show that $\begin{matrix} x = e^{4t} \\ y = e^{4t} \end{matrix}$ and $\begin{matrix} x = e^{-2t} \\ y = -e^{-2t} \end{matrix}$ are the solutions of the homogeneous system

$\begin{matrix} \frac{dx}{dt} = x + 3y \\ \frac{dy}{dt} = 3x + y \end{matrix}$ and find the particular solution $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ of the given system for which $x(0) = 5$ and $y(0) = 1$.

**Solution**- Let

$$\begin{matrix} \frac{dx}{dt} = x + 3y \\ \frac{dy}{dt} = 3x + y \end{matrix} \qquad (1)$$

First, we show that each of the pair $\begin{matrix} x = e^{4t} \\ y = e^{4t} \end{matrix}$ and $\begin{matrix} x = e^{-2t} \\ y = -e^{-2t} \end{matrix}$ satisfy the system (1). In order to determines a particular solution of (1), let us consider $\begin{matrix} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{matrix}$ (2) be a particular solution of (1), where the constants $c_1$ and $c_2$ are to be determined. Putting the values of $x_1(t) = e^{4t}$, $x_2(t) = e^{-2t}$, $y_1(t) = e^{4t}$ and $y_2(t) = -e^{-2t}$ in (2) and using the given conditions $x(0) = 5$ and $y(0) = 1$, we obtain $c_1 = 3$ and $c_2 = 2$.

Therefore $\begin{matrix} x = 3e^{4t} + 2e^{-2t} \\ y = 3e^{4t} - 2e^{-2t} \end{matrix}$ is a particular solution.

**Example** Show that $\begin{matrix} x = 3t - 2 \\ y = -2t + 3 \end{matrix}$ is a particular solution of the non-homogeneous system

$$\begin{aligned}\frac{dx}{dt} &= x + 2y + t - 1 \\ \frac{dy}{dt} &= 3x + 2y - 5t - 2\end{aligned}$$ and write the general solution of this system.

**Hint-** Let $$\begin{aligned}\frac{dx}{dt} &= x + 2y + t - 1 \\ \frac{dy}{dt} &= 3x + 2y - 5t - 2\end{aligned} \qquad (1)$$

Now $\begin{aligned}x &= 3t - 2 \\ y &= -2t + 3\end{aligned}$ will be a particular solution of the non-homogeneous system (1) if it satisfies the system (1). In order to find a general solution of system (1), we have to find a solution corresponding homogeneous system $\begin{aligned}\frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 3x + 2y\end{aligned}$ (2) to system (1) as similar in example in equation (3).

**Answer-** $$\begin{aligned}x &= 2c_1e^{4t} + c_2e^{-t} + 3t - 2 \\ y &= 3c_1e^{4t} - c_2e^{-t} - 2t + 3\end{aligned}$$

**Homogeneous Linear Systems with Constant Coefficients-** Let us consider a homogeneous linear system with constant coefficients $$\begin{aligned}\frac{dx}{dt} &= a_1x + b_1y \\ \frac{dy}{dt} &= a_2x + b_2y\end{aligned} \qquad (1)$$

Where $a_1$, $b_1$, $a_2$ and $b_2$ are constants. Suppose $$\begin{aligned}x &= Ae^{mt} \\ y &= Be^{mt}\end{aligned} \qquad (2)$$

(where $A$, $B$ and m are to be determined) be a solution of the system (1), then it satisfies (1) so

$$\begin{aligned}Ame^{mt} &= (a_1A + b_1B)e^{mt} \\ Bme^{mt} &= (a_2A + b_2B)e^{mt}\end{aligned}$$

Or

$$\begin{aligned}(a_1 - m)A + b_1B &= 0 \\ a_2A + (b_2 - m)B &= 0\end{aligned} \qquad (3)$$

is a system of equations of the form $ax=0$ has a trivial solution $x=0$, if $A=B=0$ so for a nontrivial solution $x\neq 0$ of (3), we have $a=0$ i.e

$\begin{vmatrix} a_1-m & b_1 \\ a_2 & b_2-m \end{vmatrix}=0$ , on expanding we obtain a quadratic equation in m

$$m^2-(a_1+b_2)m+(a_1b_2-a_2b_1)=0 \qquad (4)$$

gives two values of m say $m_1$ and $m_2$. Now the following three cases arise

**Case-1** If $m_1$ and $m_2$ are real and distinct, then corresponding to $m_1$, we find the values of A and B say $A_1$ and $B_1$ by equation (3), so the first nontrivial solution is $\begin{matrix} x = A_1e^{m_1 t} \\ y = B_1e^{m_1 t} \end{matrix}$ . Similarly corresponding to $m_2$, we find the another nontrivial solution $\begin{matrix} x = A_2e^{m_2 t} \\ y = B_2e^{m_2 t} \end{matrix}$

Therefore the general solution is $\begin{matrix} x = c_1(A_1e^{m_1 t})+c_2(A_2e^{m_2 t}) \\ y = c_1(B_1e^{m_1 t})+c_2(B_2e^{m_2 t}) \end{matrix}$

**Example-** Find the general solution of the system of equations

$$\frac{dx}{dt}=x+y$$
$$\frac{dy}{dy}=4x-2y$$

Solution- Let $\begin{matrix} \dfrac{dx}{dt}=x+y \\ \dfrac{dy}{dy}=4x-2y \end{matrix}$ (1)

On comparing $a_1=1, b_1=1, a_2=4$ and $b_2=-2$, the auxiliary equation is $m^2+m-6=0$ gives $m=-3,\ 2$

Where A and B satisfy $\begin{matrix} (1-m)A+B=0 \\ 4A+(-2-m)B=0 \end{matrix}$ (2)

When $m=-3$, then by (2) we get $A=1,\ B=-4$ and the first nontrivial solution is $\begin{matrix} x=e^{-3t} \\ y=-4e^{-3t} \end{matrix}$.

Similarly for $m = 2$, then by (2) we get $A = 1, B = 1$ and the another nontrivial solution is

$$\begin{aligned} x &= e^{2t} \\ y &= 4e^{2t} \end{aligned}.$$

Therefore the general solution is $\begin{aligned} x &= c_1 e^{-3t} + c_2 e^{2t} \\ y &= -4c_1 e^{-3t} + c_2 e^{2t} \end{aligned}$

**Example**- Find the general solution of the system $\begin{aligned} \frac{dx}{dt} &= -3 + 4y \\ \frac{dy}{dy} &= -2x + 3y \end{aligned}$

**Answer**- $\begin{aligned} x &= 2c_1 e^{-t} + c_2 e^{t} \\ y &= c_1 e^{-t} + c_2 e^{t} \end{aligned}$

**Case-2** If $m_1$ and $m_2$ are conjugate complex numbers of the form $a \pm ib$, where a and b are real numbers with $b \neq 0$, then we consider two linearly independent solutions $\begin{aligned} x &= A_1^* e^{(a+ib)t} \\ y &= B_1^* e^{(a+ib)t} \end{aligned}$ (1) and $\begin{aligned} x &= A_2^* e^{(a-ib)t} \\ y &= B_2^* e^{(a-ib)t} \end{aligned}$, where $A_1^* = A_1 + iA_2$, $B_1^* = B_1 + iB_2$, $A_2^* = A_1 - iA_2$ and $B_2^* = B_1 - iB_2$ resp. Putting the values of $A_1^*$ and $B_1^*$ in (1), we have

$$\begin{aligned} x &= (A_1 + iA_2)e^{at}(\cos bt + i\sin bt) \\ y &= (B_1 + iB_2)e^{at}(\cos bt + i\sin bt) \end{aligned}$$

Or

$$\begin{aligned} x &= e^{at}\left[(A_1 \cos bt - A_2 \sin bt) + i(A_1 \sin bt + A_2 \cos bt)\right] \\ y &= e^{at}\left[(B_1 \cos bt - B_2 \sin bt) + i(B_1 \sin bt + B_2 \cos bt)\right] \end{aligned}$$

Equating real and imaginary parts, we obtain two linearly independent solutions say

$$\begin{aligned} x &= e^{at}(A_1 \cos bt - A_2 \sin bt) \\ y &= e^{at}(B_1 \cos bt - B_2 \sin bt) \end{aligned} \quad (3) \quad \text{and} \quad \begin{aligned} x &= e^{at}(A_1 \sin bt - A_2 \cos bt) \\ y &= e^{at}(B_1 \sin bt - B_2 \cos bt) \end{aligned} \quad (4)$$

Therefore the general solution is

$$\begin{aligned} x &= e^{at}\left[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)\right] \\ y &= e^{at}\left[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)\right] \end{aligned}$$

**Example-** $$\frac{dx}{dt} = 4x - 2y$$ $$\frac{dy}{dt} = 5x + 2y$$

**Hint-** $$\frac{dx}{dt} = 4x - 2y \qquad \frac{dy}{dt} = 5x + 2y \qquad (1)$$

The auxiliary equation is $m^2 - 6m + 18 = 0$ gives $m = 3 \pm 3i$, taking a nontrivial solution $x = (A_1 + iA_2)e^{3t}(\cos 3t + i\sin 3t)$, $y = (B_1 + iB_2)e^{3t}(\cos 3t + i\sin 3t)$ (2) of (1), where $A_1, B_1, A_2$ and $B_2$ are to be determined. For this (2) satisfies (1) and equating the coefficients of $\cos 3t$ and $\sin 3t$ on both sides.

**Answer-** $$x = e^{3t}(2c_1\cos 3t + 2c_2\sin 3t)$$ $$y = e^{3t}\left[c_1(\cos 3t + 3\sin 3t) + c_2(\sin 3t - 3\cos 3t)\right]$$

**Case -3** If $m_1 = m_2 = m$ are equal roots then we should have only one linearly solution

$x = Ae^{mt}$, $y = Be^{mt}$ and the 2nd linearly independent solution will be of the form $x = Ate^{mt}$, $y = Bte^{mt}$. But actually, we consider the 2nd linearly independent solution

$x = (A_1 + A_2 t)e^{mt}$, $y = (B_1 + B_2 t)e^{mt}$, where $A, B, A_1, A_2$, $B_1$ and $B_2$ are to be determined.

Therefore the general solution is $$x = c_1Ae^{mt} + c_2(A_1 + A_2 t)e^{mt}$$ $$y = c_1Be^{mt} + c_2(B_1 + B_2 t)e^{mt}$$

**Example-** Find the general solution of the system

$$\frac{dx}{dt} = 3x - 4y$$
$$\frac{dy}{dt} = x - y$$

**Solution-** Let $\begin{aligned} \frac{dx}{dt} &= 3x - 4y \\ \frac{dy}{dt} &= x - y \end{aligned}$ (1)

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1$$

Let $\begin{aligned} x &= Ae^t \\ y &= Be^t \end{aligned}$ (2)

be a solution of (1), where A and B satisfy

$\begin{aligned} 2A - 4B &= 0 \\ A - 2B &= 0 \end{aligned}$ gives $A = 2,\ B = 1$,so

$$\begin{aligned} x &= 2e^t \\ y &= e^t \end{aligned} \qquad (3)$$

be a first linearly independent solution of (1). We consider the second linearly independent solution of (1) of the form $\begin{aligned} x &= (A_1 + A_2 t)e^t \\ y &= (B_1 + B_2 t)e^t \end{aligned}$ (4)

so it satisfies (1)

$\begin{aligned} (2A_1 - A_2 - 4B_1) + (2A_1 - 4B_2)t &= 0 + 0t \\ (A_1 - 2B_1 - B_2) + (A_2 - 2B_2)t &= 0 + 0t \end{aligned}$ on equating both sides we have

$\begin{aligned} 2A_1 - A_2 - 4B_1 &= 0 \\ 2A_1 - 4B_2 &= 0 \end{aligned}$ and $\begin{aligned} A_1 - 2B_1 - B_2 &= 0 \\ A_2 - 2B_2 &= 0 \end{aligned}$ (5)

On solving the equations in (5), we obtain $A_1 = 1, B_1 = 0,\ A_2 = 2$ & $B_2 = 1$

The another linearly independent solution is

$$\begin{aligned} x &= (1 + 2t)e^t \\ y &= te^t \end{aligned}$$

Therefore the general solution is

$$x = 2c_1e^t + c_2(1+2t)e^t$$
$$y = c_1e^t + c_2te^t$$

**Example-** Find the general solution of the system

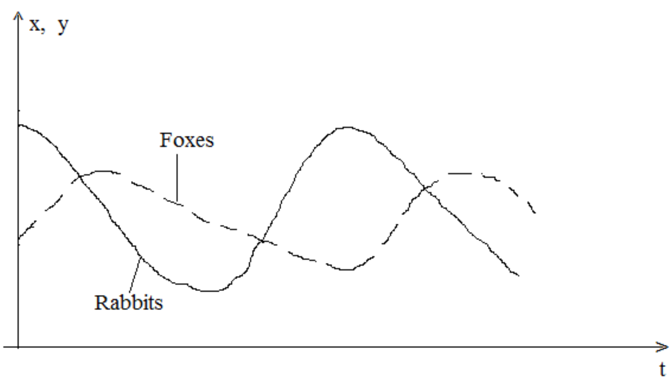
$$\frac{dx}{dt} = 5x + 4y$$
$$\frac{dy}{dt} = -x + y$$

**Answer-** $x = -2c_1e^{3t} + c_2(1+2t)e^{3t}$
$y = c_1e^{3t} - c_2te3t$

**Non-Linear Systems: Volterra's Prey- Predator Equations-**

Everyone knows that there is a constant struggle for survival among different species of animals living in the same environment. One kind of animal survives by eating another and a second by

For an example of this universal conflict between the predator and its prey, let us imagine an island inhabited by foxes and rabbits. The foxes eat rabbits and the rabbits eat clovers. Let us assume that there is so much clovers then the rabbits have an ample supply of food. When the rabbits are abundant, then the foxes flourish and their population grows. When the foxes become too numerous and eat too many rabbits, then they enter into a period of famine and their population begins to decline. As the foxes decrease, then the rabbits become relatively safe and their population starts to increase again. Thus we have an endless repeated cycle of the increase and decrease in two species of animals and the fluctuations in two species are given by the following figure




If $x$ and $y$ are the number of rabbits and foxes at any time t, then in the presence of an unlimited supply of clovers, The rate of change of rabbits is $\frac{dx}{dt} = ax,\ a > 0$, after some encounter between the rabbits and foxes the rate of change of rabbits is $\frac{dx}{dt} = ax - bxy,\ a, b > 0$ (1)

In the absence of rabbits the foxes die and the rate of change of foxes is $\frac{dy}{dt} = -cy,\ c > 0$ and after some encounter of foxes with rabbits their population grows and the rate of change of foxes become

$$\frac{dy}{dt} = -cy + dxy, \quad c, d > 0 \tag{2}$$

These two equations are called the volterra's prey-predator equations.

For the solution of these equations, we divide (2) by (1)

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-y(c - dx)}{x(a - by)}$$

Or

$$\frac{dy}{dx} = \frac{-y(c - dx)}{x(a - by)} \tag{3}$$

on separating the variables, we have

$$\frac{(c - dx)dx}{x} + \frac{(a - by)dy}{y} = 0$$

$$\int \left( \frac{c}{x} - d \right) dx + \int \left( \frac{a}{y} - b \right) dy = 0$$

On integrating, we have

$$\begin{aligned} &c \log x + a \log y = dx + by + \log K \\ &or \quad x^c y^a = K e^{(dx + by)} \end{aligned} \tag{4}$$

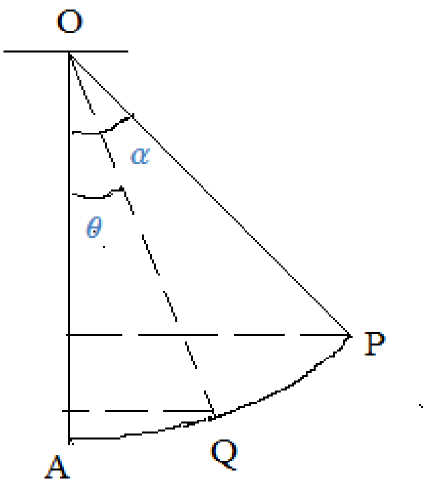
In order to determine K putting $x(t_0) = x_0,\ y(t_0) = y_0$ in (4) so

$$K = x_0^c y_0^a e^{-(d\, x_0 + b\, y_0)}$$

Therefore the solution of volterra's prey- predator equations is

$$x^c y^a = \left(x_0^c y_0^a e^{-(d\,x_0+b\,y_0)}\right)e^{(d\,x+b\,y)}$$

**Non-Linear Equations-** Let us consider the motion of a pendulum consisting a bob of mass m attached to one end of a light rod of length a. If the bob is pulled to one side through an angle $\alpha$ and then released, let $\theta$ be the position of the bob after time t s.t. AQ= s, thgn by the principle of




conservation of energy,

Gain in kinetic energy =Loss in potential energy

$$\frac{1}{2}mv^2 = mg(a\cos\theta - a\cos\alpha)$$

$$\frac{1}{2}v^2 = ga(\cos\theta - \cos\alpha) \qquad (1)$$

Also $s = a\theta$, so $v = \frac{ds}{dt} = a\frac{d\theta}{dt}$, putting in (1)

$$\frac{1}{2}a^2\left(\frac{d\theta}{dt}\right)^2 = ag(\cos\theta - \cos\alpha)$$

$$\frac{1}{2}a\left(\frac{d\theta}{dt}\right)^2 = g(\cos\theta - \cos\alpha)$$

Differentiating w. r. to t

$$\left(\frac{1}{2}a\right)\left(2\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) = g\left(-\sin\theta\frac{d\theta}{dt} - 0\right)$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{a}\sin\theta$$

Replacing $\theta$ by x, so

$$\frac{d^2x}{dt^2} + \frac{g}{a}\sin x = 0 \qquad (1)$$

is a non-linear differential equation of first order .

If x is small, then $\sin x = x$, so it becomes linear $\frac{d^2x}{dt^2} + \frac{g}{a}x = 0$, if the damping (or resistance) force is proportional to velocity, then the equation of motion is

$$\frac{d^2x}{dt^2} + \left(\frac{c}{m}\right)\frac{dx}{dt} + \frac{g}{a}\sin x = 0 \qquad (2)$$

is a non-linear differential equation of 2nd order.

Also $\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0$ (3)

is a non-linear vander pol equation.

Now, we consider a 2nd order non-linear differential equations of the form

$$\frac{d^2x}{dt^2} = f\left(x,\ \frac{dx}{dt}\right) \qquad (4)$$

**Autonomous System and Phase Plane-** Suppose a particle of unit mass moves on the x axis and $f\left(x,\ \frac{dx}{dt}\right)$ is the force acting on it, then the values of pair $\left(x(position),\ \frac{dx}{dt}(velocity)\right)$ are called the phase of the system at each instant and the plane containing $x$ and $\frac{dx}{dt}$ is called the phase plane.

Introducing the new variable $y = \frac{dx}{dt}$, then the 2nd order non-linear differential system (4) is equivalent to the system

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{d^2x}{dt^2} &= \frac{d}{dt}(y) = f(x, y)\end{aligned} \quad \text{i.e} \quad \begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= f(x, y)\end{aligned} \tag{5}$$

A system of the form $\frac{dy}{dt} = f\left(x(t),\ y(t)\right)$ is called a non-autonomous differential system and a system $\frac{dy}{dt} = f\left(x,\ y\right)$ in which the RHS does not contain the independent variable is called a autonomous differential system.

The functions $x(t)$ and $y(t)$ (where t is a parameter) are the solutions of (5) and define a curve in the x-y plane, which is also called the phase plane because $\left(x,\ y = \frac{dx}{dt}\right)$.

Now in general, we consider a system of the form

$$\begin{aligned}\frac{dx}{dt} &= F(x,\ y) \\ \frac{dy}{dt} &= G(x, y)\end{aligned} \tag{6}$$

Where $F(x, y)$ and $G(x, y)$ are continuous functions of x & y and have continuous first partial derivatives in the phase plane.

**Note-** If $t_0$ is any number and $(x_0,\ y_0)$ is any point in the phase plane the there exists a unique solution $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ of (6) with $\begin{matrix} x(t_0) = x_0 \\ y(t_0) = y_0 \end{matrix}$.

**Path-** If both the functions $x(t)$ and $y(t)$ are not constant functions, then the solution $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ of (6) is a curve in the phase plane and it is also called a path of the system.

**Note**- If $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ is a solution of (6) then $\begin{matrix} x = x(t + c) \\ y = y(t + c) \end{matrix}$ is also a solution of (6) for any constant c. Thus each path is represented by many solutions which differ from one another only by a translation of parameter.

**Critical Points-** The points $(x_0, y_0)$ at which both the functions F and G vanish i.e $F(x_0,\ y_0)=0$ and $G(x_0,\ y_0)=0$ are called the critical points of the system $\begin{matrix} \frac{dx}{dt}=F(x,\ y) \\ \frac{dy}{dt}=G(x,y) \end{matrix}$ and at such a point a unique constant solution $\begin{matrix} x=x_0 \\ y=y_0 \end{matrix}$ exists and it is not a path. Hence no path goes through a critical point.

**Isolated Critical Point-** A critical point $(x_0, y_0)$ is said to be an isolated critical point if there exists a circle centered on $(x_0, y_0)$ that contains no other critical point.

**Note-** The followings are needed to describe a phase portrait of two dimensional fluid motion,

1. The critical points.
2. The arrangement of paths near critical points.
3. The stability of critical points.
4. Closed paths.

**Example-** Describe the phase portrait of the following

1- $\begin{matrix} \frac{dx}{dt}=0 \\ \frac{dy}{dt}=0 \end{matrix}$ 2- $\begin{matrix} \frac{dx}{dt}=x \\ \frac{dy}{dt}=0 \end{matrix}$ 3- $\begin{matrix} \frac{dx}{dt}=-x \\ \frac{dy}{dt}=-y \end{matrix}$

**Solution-**1.For critical points putting each of F and G equal to zero so F(x, y)=0 and G(x, y)=0 for all values of x and y. Therefore each point of a phase plane is a critical point. For paths either we integrate separately $\frac{dx}{dt}=F(x,y)$ and $\frac{dy}{dt}=G(x,y)$ or we integrate $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{G(x,y)}{F(x,y)}$.

So on integrating $\frac{dx}{dt}=0$ & $\frac{dy}{dt}=0$ we have $x=c_1$ & $y=c_2$. Hence no path exists(because both x and y are constant functions).

**Answer 2-** every point on y axis is a critical point and paths are horizontal half linesdirected out to the left and right from y axis.

**Answer 3-** point (0, 0) is the only critical point, and paths are half lines of all possible slopes directed in toward the origen.

**Example-** Find the critical points of the following

1- $\frac{d^2x}{dt^2}+\frac{dx}{dt}-(x^3+x^2-2x)=0$

2- $\begin{aligned}\frac{dx}{dt}&=y^2-5x+6\\ \frac{dy}{dt}&=x-y\end{aligned}$

**Solution**- 1 putting $\frac{dx}{dt}=y$ in $\frac{d^2x}{dt^2}+\frac{dx}{dt}-(x^3+x^2-2x)=0$, so it becomes

$$\begin{aligned}\frac{dx}{dt}&=y\\ \frac{dy}{dt}&=(x^3+x^2-2x)-y\end{aligned} \qquad (1)$$

For critical points putting $y=0$ & $(x^3+x^2-2x)-y=0$, we have $y=0$ & $x(x^2+x-2)=0$

$x=0,1,-2$

So the critical points are $(0,0)$, $(1,0)$ & $(-2,0)$.

**Answer**-2 (2, 2) & (3, 3).

**Unit- III**

**Critical Points & Stability for Linear Systems-** Let us consider an autonomous linear system with constant coefficients

$$\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned} \qquad (1)$$

Then (1) can be written as

$$\begin{aligned} \frac{dx}{dt} &= a_1 x + b_1 y \\ \frac{dy}{dt} &= a_2 x + b_2 y \end{aligned} \qquad (2)$$

has a critical point $(0,0)$. Suppose $\begin{matrix} x = Ae^{mt} \\ y = Be^{mt} \end{matrix}$ be the solution of (2) and if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then the auxiliary equation of system (2) is

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0 \qquad (3)$$

and it is a quadratic equation in m. The nature of a critical point of (2) is determined by the nature of roots $m_1$ & $m_2$ of (3) which can be classified in two categories such as major cases and minor cases.

**Major Cases**: Case 1- The roots $m_1$ & $m_2$ are real, distinct and of same sign, then the critical point is a node i.e

$$\begin{matrix} m_1 & \neq & m_2 & \\ - & & - & (Node \,\&\, asymptotically\ stable) \\ + & & + & (Node \,\&\, unstable) \end{matrix}$$

Case 2- The roots $m_1$ & $m_2$ are real, distinct and of opposite signs, then the critical point is a saddle point i.e

$$\begin{matrix} m_1 & \neq & m_2 & \\ - & & + & (saddle \,\&\, unstable) \end{matrix}$$

Case 3- The roots $m_1$ & $m_2$ are conjugate complex numbers, but not purely imaginary, then the critical point is spiral i.e

$$m_1 \;\&\, m_2 \;=\; \begin{cases} -a+ib\,(spiral\,\&\,asymptotically\;\;stable) \\ a-ib\,(spiral\;\&\,unstable) \end{cases}$$

and $\dfrac{d\theta}{dt}=\pm$

**Borderline cases:** Case 4**-** The roots $m_1$ & $m_2$ are real and equal, then the critical point is a node i.e $m_1=m_2=m<0\,(node\,\&\,asymptotically\;\;stable)$ and if

$m_1=m_2=m>0\,(node\;\&\;un\,stable)$ .

Case 5- The roots $m_1$ & $m_2$ are purely imaginary, then the critical point is a centre and stable, but not asymptotically stable.

**Example-** For each of the given linear system , find (1) Critical points (2) find the general solution (3) find the differential equation of paths (4) solve the equation of paths (5) sketch a few of the paths (6) discuss the stability of critical points.

a- $\begin{aligned}\dfrac{dx}{dt}&=-x\\ \dfrac{dy}{dt}&=-2y\end{aligned}$ b- $\begin{aligned}\dfrac{dx}{dt}&=4y\\ \dfrac{dy}{dt}&=-x\end{aligned}$

**Solution:** a- Here (1) the critical point is $(0,\,0)$, (2) solution is $\begin{aligned}x&=c_1e^{-t}\\ y&=c_2e^{-2t}\end{aligned}$ , (3) $\dfrac{dy}{dx}=\dfrac{2y}{x}$,

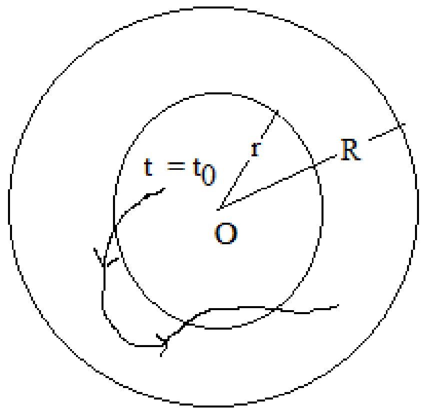
(4) $y=c\,x^2$, (5) paths are exponential curves, (6) since both the limits $\lim_{t\to\infty} x(t)\;\; and\;\; \lim_{t\to\infty} y(t)$ exist, then the critical point is asymptotically stable.

**Note-** If $\lim_{t\to\infty}$ both $x(t)$ and $y(t)$ exist, then the critical point is asymptotically stable and if $\lim_{t\to\infty}$ one of these $x(t)$ and $y(t)$ exist, then the critical point is unstable.

Solution b- (1) $(0,\,0)$, (2) $\begin{aligned}x&=2c_1\cos 2t+2c_2\sin 2t\\ y&=-c_1\sin 2t+c_2\cos 2t\end{aligned}$, (3) $\dfrac{dy}{dx}=-\dfrac{x}{4y}$, (4) $\dfrac{x^2}{4c^2}+\dfrac{y^2}{c^2}=1$, (5) stable but not asymptotically stable.

**Note- Stable and unstable critical points**: A critical point is said to be stable, if for each positive number R there exists a positive number $r\le R$ such that every path which is inside the circle $x^2+y^2=r^2$ for some $t=t_0$ remains inside the circle $x^2+y^2=R^2$ for some $t>t_0$. A critical point is said to be asymptotically stable, if it is stable and there exists a circle

$x^2 + y^2 = r_0^2$ such that every path which is inside this circle for some $t = t_0$, approaches the origin as $t \to \infty$ and if the critical point is not stable, then it is called unstable.




**Theorem-** The critical point $(0,0)$ of the linear system $\begin{aligned}\frac{dx}{dt} &= a_1x + b_1y \\ \frac{dy}{dt} &= a_2x + b_2y\end{aligned}$ is stable if and only if the roots of the auxiliary of the given system have non positive real parts, and it is asymptotically stable if and only if both roots have negative real parts.

Proof- The auxiliary equation of the given system can be written as

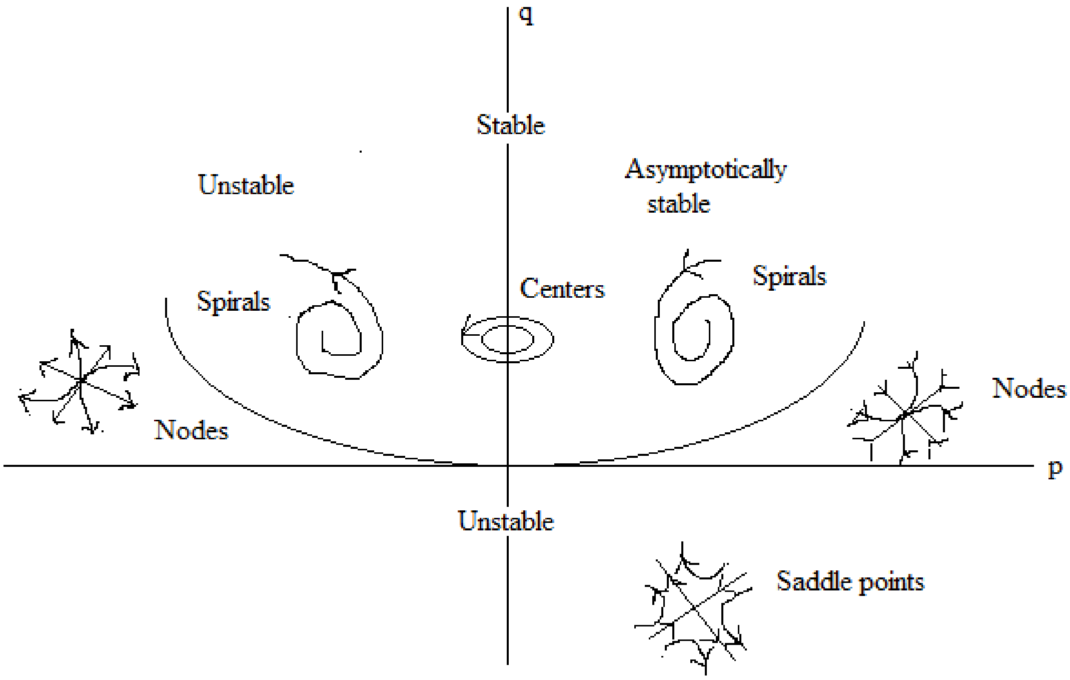
$$(m - m_1)(m - m_2) = m^2 + pm + q = 0 \tag{1}$$

Where $p = -(m_1 + m_2)$ & $q = m_1m_2$, let us consider p and q axes and excluding the case $q = 0$, then by (1), we have $m_1, m_2 = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$

Now the three cases arise $\sqrt{p^2 - 4q} = 0, + \; or \; -$

If $p^2 - 4q = 0$, i.e on the parabola $p^2 - 4q = 0$ ,them the roots $m_1$ & $m_2$ are real and equal and the critical points are node.

If $p^2 - 4q < 0$, then the roots $m_1$ & $m_2$ are complex conjugate numbers, the critical points are spirals and centers.




If $p^2 - 4q > 0$, then the roots $m_1$ & $m_2$ are real, distinct and the critical points are saddle points. If $p^2 - 4q > 0$ and $q < 0$, then the roots $m_1$ & $m_2$ are real and distinct, then the critical points are spirals.

**Theorem-** The critical point $(0, 0)$ of the linear system $\begin{aligned}\frac{dx}{dt} &= a_1x + b_1y \\ \frac{dy}{dt} &= a_2x + b_2y\end{aligned}$ is asymptotically stable if and only if the coefficients $p = -(a_1 + b_2)$ & $q = (a_1b_2 - a_2b_1)$ of the auxiliary equation are both positive.

**Example-** Determine the nature and stability properties of the critical point $(0,0)$ for each of the following linear autonomous systems

1. $\begin{aligned}\frac{dx}{dt} &= -x - 2y \\ \frac{dy}{dt} &= 4x - 5y\end{aligned}$ 2. $\begin{aligned}\frac{dx}{dt} &= -3x + 4y \\ \frac{dy}{dt} &= -2x + 3y\end{aligned}$ 3. $\begin{aligned}\frac{dx}{dt} &= 4x - 3y \\ \frac{dy}{dt} &= 8x - 6y\end{aligned}$

**Solution** 1- Let $\begin{aligned}\frac{dx}{dt} &= -x-2y \\ \frac{dy}{dt} &= 4x-5y\end{aligned}$

For critical points solving $\begin{aligned}-x-2y=0 \\ 4x-5y=0\end{aligned}$, we have $(x, y) = (0,0)$

The auxiliary equation is $m^2 - (-1-5)m + (5+8) = 0$ i.e $m^2 + 6m + 13 = 0$

i.e $m = -3 \pm 2i$, since $m_1$ & $m_2$ are complex conjugate numbers $a \pm ib$ & $a < 0$, then the critical point are spirals and asymptotically stable.

**Answer 2-** Unstable saddle point

**Answer 3-** The critical point is not isolated.

**Stability by Liapunov's Direct Method-** Liapunov's direct method is used for studying the stability problems of a linear autonomous system $\begin{aligned}\frac{dx}{dt} &= F(x, y) \\ \frac{dy}{dt} &= G(x, y)\end{aligned}$ (1) with an isolated critical point $(0,0)$. Let $C = [x(t), y(t)]$ be any path of (1) and $E(x, y)$ be any function which is continuous and has continuous first partial derivatives in a region containing this path. If a point moves along this path with $x = x(t)$ & $y = y(t)$, then $E(x, y)$ can be regarded as a function of t and the rate of change of $E(x, y)$ with respect to t is

$$\begin{aligned}\frac{dE}{dt} &= \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G\end{aligned} \qquad (2)$$

is called a Liapunov's direct method.

**Positive Definite-** A function $E(x, y)$ is said to be positive definite, if $E(0,0) = 0$ & $E(x, y) > 0 \quad \forall\ (x, y) \neq (0,0)$.

**Negative Definite-** A function $E(x, y)$ is said to be negative definite, if $E(0,0) = 0$ & $E(x, y) < 0 \quad \forall\ (x, y) \neq (0,0)$.

**Positive Semi Definite**- A function $E(x, y)$ is said to be positive semi definite, if $E(0,0) = 0$ & $E(x, y) \geq 0 \quad \forall\ (x, y) \neq (0,0)$.

**Negative Semi Definite-** A function $E(x,y)$ is said to be negative semi definite, if $E(0,0)=0$ & $E(x,y)\le 0\quad \forall\,(x,y)\ne(0,0)$.

**Note-** Let us consider a positive definite function $E(x,y)$ of the form

$E(x,y)=ax^{2m}+by^{2n}$ ,where $a$ & $b$ are positive constants and $m$ & $n$ are positive integers.

**Liapunov's Function-** A positive definite function $E(x,y)$ is said to be a Liapunov function if $\frac{dE}{dt}=\frac{\partial E}{\partial x}F+\frac{\partial E}{\partial y}G$ is negative semi definite i.e $\frac{dE}{dt}\le 0$, so it is a decreasing function.

**Theorem-** The function $E(x,y)=ax^2+bxy+cy^2$ is positive definite, if and only if $a>0$ & $b^2-4ac<0$, and is negative definite if and only if $a<0$ & $b^2-4ac<0$ and is neither if $a>0$ & $b^2-4ac>0$ *or* $a<0$ & $b^2-4ac>0$

**Example-** Determine whether each of the following function is positive definite, negative definite or neither:

A- $x^2-xy-y^2$ B- $2x^2-3xy+3y^2$ C- $-2x^2+3xy-y^2$ D- $-x^2-4xy-5y^2$

**Solution** A- Here $E(x,y)=x^2-xy-y^2$ comparing $a=1>0$ & $b^2-4ac=5>0$ so it is neither.

**Answer B-** Here $E(x,y)=2x^2-3xy+3y^2$ comparing $a=2>0$ & $b^2-4ac=-15<0$, so it positive definite.

**Answer C**- Here $E(x,y)=-2x^2+3xy-y^2$ on comparing $a=-2<0$ & $b^2-4ac=1>0$, so it is neither.

**Answer D**- Here $E(x,y)=-x^2-4xy-5y^2$ on comparing $a=-1<0$ & $b^2-4ac=-4<0$, so it is negative definite.

**Example** – Show that $(0,\,0)$ is an asymptotically stable critical point for each of the following systems:

A- $$\frac{dx}{dt}=-3x^3-y,\quad \frac{dy}{dt}=x^5-2y^3$$

B- $$\frac{dx}{dt}=-2x+xy^3,\quad \frac{dy}{dt}=-x^2y^2-y^3$$

**Solution**- Let $\begin{aligned}\frac{dx}{dt}&=-3x^3-y\\ \frac{dy}{dt}&=x^5-2y^3\end{aligned}$ (1)

On comparing $F(x,y)=-3x^3-y$ & $G(x, y)=x^5-2y^3$, Now we consider $E(x,y)=ax^{2m}+by^{2n}$ where $a,b,m\&n$ are to be determined. The critical point $(0, 0)$ will be stable, if $\frac{dE}{dt}=\frac{\partial E}{\partial x}F+\frac{\partial E}{\partial y}G$, is negative semi definite i.e

$$\frac{dE}{dt}=2amx^{2m-1}(-3x^3-y)+2bny^{2n-1}(x^5-2y^3)$$

$\frac{dE}{dt}=-6amx^{2m+2}-2amx^{2m-1}y+2bnx^5y^{2n-1}-4bny^{2n+2}$ will be negative semi definite if

$-2amx^{2m-1}y+2bnx^5y^{2n-1}=0$ *i.e* $2amx^{2m-1}y=2bnx^5y^{2n-1}$

equating the powers of $x$ & $y$ and also the coefficients of xy on both sides, we have

$2m-1=5\Rightarrow m=3$ & $2n-1=1\Rightarrow n=1$ and $6a=2b\Rightarrow a=1\&b=3$

So $E(x,y)=x^6+3y^2$.

Similarly 2nd part can be solved

**Answer** 2- $E(x,y)=x^2+y^2$

**Example**- Show that $(0, 0)$ is an unstable critical point for the system

$$\frac{dx}{dt}=2xy+x^3$$
$$\frac{dy}{dt}=-x^2+y^5$$

**Solution**- Let us assume contradictory that $(0, 0)$ is a stable critical point so

$\frac{dE}{dt}=4amx^{2m}y+2amx^{2m+2}-2bnx^2y^{2n-1}+2bny^{2n+4}$ will be negative semi definite if

$4amx^{2m}y=2bnx^2y^{2n-1}\Rightarrow m=n=1\ \&\ a=1\ \&b=2$, for these values of $m,n,a\&b$, the value of

$\frac{dE}{dt} > 0$ Hence the critical point origin is unstable.

**Green Function-** Let us consider an nth order linear homogeneous differential equation $L(y) = 0$ (1)

where L is a differential operator given by

$$L(y) = p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \qquad + p_n(x) \tag{2}$$

where $p_0(x),\ p_1(x), \qquad p_n(x)$ are continuous functions on $[a, b]$ and the boundary conditions are

$$V_k(y) = 0, \qquad \forall\ k = 1,\ 2, \qquad n \tag{3}$$

Where $V_k(y) = \alpha_k y(a) + \alpha_k^{(1)} y'(a) + \alpha_k^{(2)} y''(a) + \qquad + \alpha_k^{(n-1)} y^{(n-1)}(a) + \beta_k y(b) + \beta_k^1 y'(b)$

$$+ \beta_k^{(2)} y''(b) + \qquad + \beta_k^{(n-1)} y^{(n-1)}(b) \tag{4}$$

Where the linear forms $V_1, V_2 ........ \quad V_n$ in $y(a), y'(a)..... \quad y^{(n-1)}(a), y(b),\ y'(b)....... \quad y^{(n-1)}(b)$ are linearly independent.

Suppose the homogeneous boundary value problem given by (1) to (4) has only a trivial solution $y(x) = 0$. The Green function of the boundary value problem (1) to (4) is the function $G(x,t)$ constructed for any point t in $a < t < b$ and has the following properties

1- In each of the interval $[a,\ t)$ & $(t,\ b]$ the function $G(x,t)$ is considered as a function of x and is a solution of (1) i.e $L(G) = 0$. (5)

2- $G(x,t)$ is continuous and has continuous partial derivatives with respect to x up to $(n-2)$ orders on $a \le x \le b$.

3- The $(n-1)$ th derivative of $G(x,t)$ with respect to x at $x = t$ has a discontinuity of Ist kind and the jump being equal to $\left(-\frac{1}{p_0(t)}\right)$ i.e

$$\left(\frac{\partial^{n-1} G}{\partial x^{n-1}}\right)_{x=t+0} - \left(\frac{\partial^{n-1} G}{\partial x^{n-1}}\right)_{x=t-0} = -\left(\frac{1}{p_0(t)}\right)$$

4- $G(x,t)$ satisfies the boundary conditions $V_k(G) = 0, \qquad k = 1, 2.... \qquad n$.

**Note**- If the boundary value problem ( 1) to (4) has only a trivial solution $y(x) = 0$, then the operator L has a unique Green function $G(x,t)$.

**Example**- Find the Green function of the following boundary value problems

1- $y'' = 0, \quad y(0) = y(l) = 0$ 2- $y'' + \mu^2 y = 0, \quad y(0) = y(1) = 0$

3- $xy'' + y' - \left(\frac{1}{x}\right) y = 0, \quad y(0)\, is\ \ finite\ \ \&\ \ y(1) = 0$

**Solution 1**- Let $y'' = 0 \quad y(0) = y(l) = 0$ (1)

First, we show that (1) has a zero or trivial solution.

For this integrating (1) twice with respect to x, then the solution of (1) is

$y = a\,x + b$ (2)

Using the boundary conditions $y(0) = 0 \Rightarrow b = 0$ and $y(l) = 0 \Rightarrow 0 = al \Rightarrow a = 0$

Hence $y(x) = 0$ is a zero or trivial solution of (1). So (1) has a Green function

$$G(x,t) = \begin{cases} a_1 x + a_2 & 0 \le x < t \\ b_1 x + b_2 & t < x \le l \end{cases}$$

Now $G(x,t)$ satisfies the following properties

(1) $G(x,t)$ is continuous at $x = t$ so $b_1 t + b_2 = a_1 t + a_2 \Rightarrow (b_1 - a_1)t = (a_2 - b_2)$ (3)

(2) $G(x,t)$ has a discontinuity of magnitude $\frac{1}{p_0(t)}$ i.e

$$\left(\frac{\partial G}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G}{\partial x}\right)_{x=t-0} = -1 \Rightarrow b_1 - a_1 = -1 \quad (4)$$

(3) $G(x,t)$ satisfies the boundary conditions $G(0,t) = 0 \Rightarrow a_1.0 + a_2 = 0 \Rightarrow a_2 = 0$ (5)

$G(l,t) = 0 \Rightarrow b_1 l + b_2 = 0$ (6)

From (3) & (4) $a_2 - b_2 = -t$ (7)

Solving (3), (4), (5), (6) & (7) we have $b_2 = t,\ b_1 = -\frac{t}{l},\ a_1 = 1 - \frac{t}{l}$

Therefore $G(x,t) = \begin{cases} \frac{x(l-t)}{l} & 0 \le x < t \\ \frac{t(l-x)}{l} & t < x \le l \end{cases}$

**Answer 2- Solution** $a_1 = 0,\ b_1 = \frac{\sin \mu t}{\mu},\ a_2 = -\frac{\sin \mu(t-1)}{\mu \sin \mu}\ \&\ b_2 = -\frac{\sin \mu t \cos \mu}{\mu \sin \mu}$

$$G(x,t)=\begin{cases} -\dfrac{\sin\mu(t-1)\sin\mu x}{\mu\sin\mu} & 0\le x<t \\ -\dfrac{\sin\mu t\sin\mu(t-1)}{\mu\sin\mu} & t<x\le 1 \end{cases}$$

**Answer 3** Let $xy''+y'-\dfrac{1}{x}y=0 \qquad y(0)=finite \qquad y(1)=0$ (1)

**Solution-** (1) can be written as $x^2y''+xy'-y=0 \qquad y(0)=finite=k(say) \qquad y(1)=0$ (2)

Putting $x=e^u$ & $x\dfrac{d}{dx}=\dfrac{d}{du}=D,\ x^2\dfrac{d^2}{dx^2}=D(D-1)$ in (2), we have

$(D^2-1)y=0$ (3)

The solution of (3) is

$$y=ax+\frac{b}{x}$$

$$G(x,\,t)=\begin{cases} \left(\dfrac{x}{2}\right)\left(\dfrac{1-t}{t}\right) & 0\le x<t \\ \left(\dfrac{t}{2}\right)\left(\dfrac{1-x^2}{x}\right) & t<x\le 1 \end{cases}$$

**Sturm's Comparison theorem**: Let us consider two system of 2nd order differential equations $\left(p_1(t)u'\right)'+q_1(t)u=0$ (1) & $\left(p_2(t)u'\right)'+q_2(t)u=0$ (2) and assume that

1- $p_1(t), p_2(t), q_1(t)$ & $q_2(t)$ are continuous on the $[a,b]$.
2- $u_1(t)$ & $u_2(t)$ are the non trivial solutions of the given system of equations.
3- equation (2) is a sturm majorant of (1) on $[a,b]$.
4- the inequality $\dfrac{p_1(a)u_1'(a)}{u_1(a)}\ge\dfrac{p_2(a)u_2'(a)}{u_2(a)}$ holds, and the LHS & RHS become infinite when $u_1(a)=0$ & $u_2(a)=0$.
5- $u_1(t)$ has exactly $n\ge 1$ zeros at $t=t_1,t_2.........t_n$ where $t_1<t_2<..........<t_n$ of $(a,\ b]$.

Then the solution $u_2(t)$ has at least n-zero on $(a,\ t_n]$.

**Proof**: Let us define a pair of continuous functions say $\phi_1(t)$ & $\phi_2(t)$ by

$$\phi_1(t) = \tan^{-1}\left(\frac{u_1(t)}{p_1(t)u_1^{'}(t)}\right) \qquad (1)$$

$$\phi_2(t) = \tan^{-1}\left(\frac{u_2(t)}{p_2(t)u_2^{'}(t)}\right) \qquad (2)$$

on the $[a,b]$ for $0 \le \phi_i < \pi, \quad \forall\ i = 1,\ 2$ (only principle values).

Since $p_1(t) \ge p_2(t) \Rightarrow \frac{1}{p_1(t)} \le \frac{1}{p_2(t)} \Rightarrow \frac{1}{p_1(t)u_1^{'}(t)} \le \frac{1}{p_2(t)u_2^{'}(t)}$ so for a particular value at $t = a$, we have $\frac{u_1(a)}{p_1(a)u_1^{'}(a)} \le \frac{u_2(a)}{p_2(a)u_2^{'}(a)}$ hence $0 \le \phi_1(a) \le \phi_2(a) < \pi$

Also by prufer's transformation

$$\phi_1^{'}(t) = \frac{1}{p_1(t)}\cos^2\phi_1 + q_1(t)\sin^2\phi_1 \qquad (3)$$

$$\phi_2^{'}(t) = \frac{1}{p_2(t)}\cos^2\phi_2 + q_2(t)\sin^2\phi_2 \qquad (4)$$

Putting $f_i(t,\ \phi) = \frac{1}{p_i(t)}\cos^2\phi_i + q_i(t)\sin^2\phi_i, \qquad \forall\ i = 1,\ 2$ using in (3) & (4), we have

$$\phi_1^{'}(t) = f_1(t,\phi) \ \ \&\ \ \phi^{'}_2(t) = f_2(t,\phi) \Rightarrow \phi_i^{'} = f_i(t,\phi) \quad \forall\ i = 1, 2 \qquad (5)$$

The solution of (3) is $\phi_1(t)$ with condition $(a,\ \phi_1(a))$ and the solution of (4) is $\phi_2(t)$ with condition $(a,\ \phi_2(a))$.

Since $p_1(t) \ge p_2(t) \Rightarrow \frac{1}{p_1(t)} \le \frac{1}{p_2(t)} \quad \forall t \in [a,\ b]$ so by (5), we have

$\{\phi_1^{'} = f_1(t,\phi)\} \le \{\phi_1^{'} = f_2(t,\phi)\}$ on integrating, we have $\phi_1(t) \le \phi_2(t) \quad \forall t \in [a,\ b]$ so putting $t = t_n$ we have $\phi_1(t_n) \le \phi_2(t_n) \Rightarrow\ n\pi \le \phi_2(t_n)$ since $\phi_1(t)$ has exactly n zeros on $(a,b]$. This implies $u_2(t)$ has at least n zeros on $(a,\ t_n]$ (where $t_n = b$).

**Corollary:** The zeros of two linearly independent solutions $u_1(t)$ & $u_2(t)$ of $\left(p_1(t)u^{'}\right)^{'} + q_1(t)u = 0$ interlace, i.e between any two consecutive zeros of one solution there lies a zero of the other solution.

**Proof**: Let $u_1(t)$ & $u_2(t)$ be any two solutions of $\left(p_1(t)u'\right)' + q_1(t)u = 0$ (1) then both $u_1(t)$ & $u_2(t)$ satisfy (1) so

$$\left(p_1(t)u_1'(t)\right)' + q_1(t)u_1(t) = 0 \qquad (2)$$

$$\left(p_1(t)u_2'(t)\right)' + q_1(t)u_2(t) = 0 \qquad (3)$$

Let $t_1$ & $t_2$ be any two consecutive zeros of $u_1(t)$ so $u_1(t_1) = 0 = u_1(t_2)$.

Subtracting on multiplying (2) by $u_2$ and (3) by $u_1$ and then integrating between the limits $t_1$ *to* $t_2$, we have

$$\left[p_1(t)\left\{u_1'(t)u_2(t) - u_2'(t)u_1(t)\right\}\right]_{t_1}^{t_2} = 0$$

Or

$$\left[p_1(t_2)\left\{u_1'(t_2)u_2(t_2) - u_2'(t_2)u_1(t_2)\right\}\right] - \left[p_1(t_1)\left\{u_1'(t_1)u_2(t_1) - u_2'(t_1)u_1(t_1)\right\}\right] = 0$$

putting $u_1(t_1) = 0 = u_1(t_2)$, so it become

$$p_1(t_2)u_1'(t_2)u_2(t_2) = p_1(t_1)u_1'(t_1)u_2(t_1).$$

Since $u_1'(t_1)$ & $u_1'(t_2)$ are of opposite signs so $u_2(t_1)$ & $u_2(t_2)$ will be of opposite signs. Hence $u_2(t)$ has at least one zero between $t_1$ & $t_2$.