

UNIT - 4

Numerical Solution of Ordinary Differential equl.

Branch- EC 2nd year

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Numerical Solutions of Ordinary Differential Equations

Here we shall see the Numerical solutions of ordinary differential equations by various methods.

1. Taylor's series method -:

Let us consider the first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

Under the condition $y(0) = y_0$ for $x=0$

$$y = y_0 + (x-x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \dots$$

Ex-1. Use Taylor's series method to solve the equation

$$\frac{dy}{dx} = -xy, \quad y(0)=1$$

Sol. we have $\frac{dy}{dx} = -xy$

$$y' = -xy \quad \text{--- (1)}$$

On putting $x=0$ and $y=1$ in eq. (1), we get

$$y'(0) = -0 \times 1 = 0$$

$$y'' = -xy' - y \Rightarrow y''(0) = 0 - 1 = -1$$

$$y''' = -xy'' - 2y' \Rightarrow y'''(0) = 0 - 2(0) = 0$$

Taylor series expansion about $x=0$

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \dots \quad \text{--- (2)}$$

On substituting the value of $y(0)$, $y'(0)$, $y''(0)$ & $y'''(0)$ in eq. (2) we get

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots \quad \text{Ans}$$

let us consider the first order differential equation.

$$\frac{dy}{dx} = f(x, y)$$

and $y = y_0$ for $x = x_0$.

$$y_{n+1} = y_0 + \int_{x_0}^x f(x, y_n) dx$$

Example Using Picard's method, find a solution of $\frac{dy}{dx} = 1 + xy$ upto the third approximation, when $x_0 = 0, y_0 = 0$

Sol.

$$\frac{dy}{dx} = 1 + xy = f(x, y)$$

$$y_1 = y_0 + \int_0^x (1 + xy_0) dx = 0 + \int_0^x (1 + 0) dx$$

$$y_1 = x$$

$$y_2 = y_0 + \int_0^x (1 + xy_1) dx = 0 + \int_0^x (1 + x^2) dx = x + \frac{x^3}{3}$$

$$y_3 = y_0 + \int_0^x (1 + xy_2) dx$$

$$= 0 + \int_0^x \left[1 + x \left(x + \frac{x^3}{3} \right) \right] dx$$

$$= \int_0^x \left(1 + x^2 + \frac{x^4}{3} \right) dx$$

$$y_3 = x + \frac{x^3}{3} + \frac{x^5}{15}$$

Ans.

Euler's method -

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Consider the equation

$$\frac{dy}{dx} = f(x, y)$$

When $y = y_0$ and $x = x_0$

$$y_{n+1} = y_n + h f(x_n, y_n) \quad \text{--- (1)}$$

Put $n=0$

$$y_1 = y_0 + h f(x_0, y_0)$$

Similarly $n=1, 2, 3, \dots$ in eq. (1) we get

$$y_2 = y_1 + h f(x_1, y_1)$$

Example - : Using Euler's method solve the differential equation in 4 steps :

Sol. Here $\frac{dy}{dx} = x + y = f(x, y)$

$$y(0) = 0 \quad x_0 = 0 \quad h = 0.2$$

Euler's method.

$$y_{n+1} = y_n + h f(x_n, y_n)$$

Put $n=0$

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= 0 + 0.2 [x_0 + y_0]$$

$$y_1 = 0 + 0.2 (0 + 0) = 0$$

$$\boxed{y_1 = 0}$$

$$y_2 = y_1 + h(x_1 + y_1)$$

$$= 0 + 0.2(0.2 + 0) = 0.04$$

$$x_1 = x_0 + h$$

$$= 0 + 0.2 = 0.2$$

$$y_3 = y_2 + h(x_2 + y_2)$$

$$= 0.04 + 0.2(0.4 + 0.04)$$

$$= 0.128$$

$$x_2 = x_1 + h$$

$$= 0.2 + 0.2$$

$$= 0.4$$

Simi $y_4 = 0.2736$ $y_5 = 0.48832$

$y_6 = 0.7859$ Ans.

Runge-Kutta formula (fourth order)

$$y = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

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Example - Use the Runge Kutta fourth order method to find $y(0.2)$ with $h=0.1$ for the initial value problem

$$\frac{dy}{dx} = \sqrt{x+y}, \quad y(0)=1$$

Sol. Here $h=0.1$ $x_0=0, y_0=1, f(x,y) = (x+y)^{1/2}$

$$K_1 = h f(x_0, y_0) = 0.1 \sqrt{0+1} = 0.1$$

$$\begin{aligned} K_2 &= h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1 f(0+0.05, 1+0.05) \\ &= 0.1 f(0.05, 1.05) \\ &= 0.1 \sqrt{0.05+1.05} = 0.10488 \end{aligned}$$

$$\begin{aligned} K_3 &= h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) \\ &= 0.1 f(0+0.05, 1+0.05244) = 0.1 f(0.05, 1.05244) \\ &= 0.1 \sqrt{0.05+1.0524} = 0.104997 \end{aligned}$$

$$\begin{aligned} K_4 &= h f(x_0 + h, y_0 + K_3) = 0.1 f(0+0.1, 1+0.10499) \\ &= 0.1 \sqrt{0.1+1.10499} = 0.10977 \end{aligned}$$

According to Runge - Kutta formula

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6} (0.1 + 0.209 + 0.209 + 0.109) = 1.1049$$

for the second step $x_0=0.1, y_0=1.1049, h=0.1$

$$K_1 = h f(x_0, y_0) = 0.1 \sqrt{0.1+1.1049} = 0.1097$$

$$\text{81mi } K_2 = 0.1144, K_3 = 0.11454, K_4 = 0.1191$$

$$y_{0.2} = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1.1049 + \frac{1}{6} (0.109 + 0.2288 + 0.229 + 0.119)$$

$$= 1.2194$$

Ans

Milne's Predictor-Corrector method -:

⑥

$$\frac{dy}{dx} = f(x, y) \quad x = x_0 + nh$$

$$y(x_0) = y_0$$

Step-1. $y_1 = y(x_0 + h), y_2 = y(x_0 + 2h), y_3 = y(x_0 + 3h)$

Step-2 $f_0 = f(x_0, y_0), f_1 = f(x_0 + h, y_1)$
 $f_2 = f(x_0 + 2h, y_2), f_3 = f(x_0 + 3h, y_3)$

$$y_4 = y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3]$$

Step-3 $f_4 = f(x_0 + 4h, y_4)$

Step-4 $y_4 = y_2 + \frac{h}{3} [f_2 + f_3 + f_4]$

Example Apply Milne's method to find a solution of the differential equation $y' = x - y^2$ in the range $0 \leq x \leq 1$ for the boundary condition $y = 0$ at $x = 0$.

Sol. we have $\frac{dy}{dx} = x - y^2, \quad x = 0, y = 0$

find the value of y_1, y_2, y_3 by Picard's method

$$y_1 = \frac{x^2}{2} = 0.020$$

$$x_1 = x_0 + h$$

$$y_2 = \frac{x^2}{2} - \frac{x^5}{20} = 0.0795$$

$$x_2 = x_1 + h$$

$$y_3 = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400} = 0.1762$$

$$x_3 = x_2 + h$$

$$f_1 = 0.1996, \quad f_2 = 0.3937, \quad f_3 = 0.5690$$

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Using the Predictor,

$$y_4 = y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3]$$

$$= 0 + \frac{4(0.2)}{3} [2(0.199) - 0.3937 + 2(0.569)]$$

$$= 0.3049$$

$$f_4 = x - y^2 = \{0.8 - (0.3049)^2\}$$

$$= 0.7070$$

By the Corrector,

$$y_4 = y_2 + \frac{h}{3} [f_2 + 4f_3 + f_4]$$

$$= 0.3046$$

$$f_4 = (x - y^2) = \{(0.8) - (0.3046)^2\} = 0.70272$$

Again using the corrector

$$y_4 = y_2 + \frac{h}{3} [f_2 + 4f_3 + f_4]$$

$$= 0.3046$$

Using Predictor

$$y_5 = y_1 + \frac{4h}{3} [2f_2 - f_3 + 2f_4]$$

$$= 0.4554$$

$$f_5 = 0.7926$$

$$\boxed{y_5 = 0.4556} \quad \leftarrow$$

$$f_5 = 0.7926$$

By corrector.

$$y_5 = y_3 + \frac{h}{3} [f_3 + 4f_4 + f_5] = \underline{0.4556}$$

$$\boxed{y(\text{at } x=1) = 0.4556} \quad \text{Ans}$$

Adam - Bash forth Predictor-Corrector method

Predictor $y_{n+1} = y_n + \frac{h}{24} [55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$

Corrector $y_{n+1} = y_n + \frac{h}{24} [9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$

Example - $y + \frac{dy}{dx} = x^2(1+y)$ and $y(1)=1, y(1.1)=1.233$

$y(1.2)=1.548, y(1.3)=1.979$. then find $y(1.4)$ by Adams - Bash forth method.

Sol. we have $\frac{dy}{dx} = x^2(1+y) = f(x, y)$

Let us find out f_0, f_{-1}, f_{-2} and f_{-3} .

$x_{-3} = 1.0$	$y_{-3} = 1.000$	$f_{-3} = (1)^2(1+1.00) = 2$
$x_{-2} = 1.1$	$y_{-2} = 1.233$	$f_{-2} = 2.702$
$x_{-1} = 1.2$	$y_{-1} = 1.548$	$f_{-1} = 3.669$
$x_0 = 1.3$	$y_0 = 1.9$	$f_0 = 5.035$

By Predictor formula

$$y_1 = y_0 + \frac{h}{24} [55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}]$$
$$= 2.572$$

$$x_1 = 1.4, y_1 = 2.572, f_1 = 7.001$$

By Corrector formula $y_1 = y_0 + \frac{h}{24} [9f_1 + 19f_0 - 5f_{-1} + f_{-2}]$

$$y_1 = 2.575$$

Ans.