UNIT - 3

Numerical Integration

Branch- EC. 2^{nd} year

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Numerical Integration

Numerical Integration is the Process of evaluating a definite integral from the values of the function given in tabular form.

Newton-Cotes Quadrature formula -:

Let $I = \int_a^b f(x)\,dx$

Let $x_0 = a$, $x_1 = a+h$, $x_2 = a+2h$, -- $x_n = a+nh = b$ and Corresponding values of $f(x)$ be y_0, y_1, y_2 ---y_n respectively.

$$P = \frac{x - x_0}{h}$$

$$\frac{dP}{dx} = \frac{1}{h} = dx = h\,dP$$

Newton's forward interpolation formula -:

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots$$

Integrating

$$\int_{x_0}^{x_0+nh} y\,dx = \int_0^n \left[y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \cdots\cdots \right] h\,dP$$

$$= h\left[Py_0 + \frac{P^2}{2!}\Delta y_0 + \left(\frac{P^3}{6} - \frac{P^2}{4}\right)\Delta^2 y_0 + \cdots \right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{6} - \frac{n^2}{4}\right)\Delta^2 y_0 + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + \right] \quad (1)$$

This is Newton-Cotes Quadrature formula.

Trapezoidal rule :- on taking $n=1$

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

Simpson's one-third rule :- on putting $n=2$ in eq (1)

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+\cdots+y_{n-2}) + 4(y_1+y_3+y_5+\cdots+y_{n-1})\right]$$

Simpson's $\frac{3}{8}$ rule :- on putting $n=3$ in eq (1)

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+\cdots+y_{n-1})+2(y_3+y_6+y_9+\cdots+y_{n-3})\right]$$

Weddle's rule :- on putting $n=6$ in eq (1)

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

Romberg Integration :-

$$I_{k,J} = I_{k-1,J+1} + \frac{I_{k-1,J+1}-I_{k-1,J}}{4^{k-1}-1}, \quad k \geq 2$$

where k = order of integration

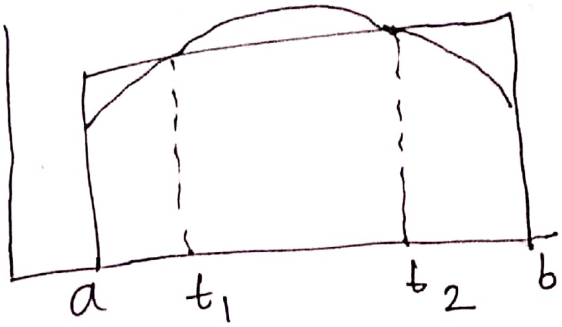
$k=1$ values for regular trapezoidal rule.

$k=2$, value using error

J = more / less accurate estimate of integral

Gaussian Quadrature :

Coordinate transformation from [a,b] to [-1,1]



$$t = \frac{b-a}{2} x + \frac{b+a}{2}$$

$x = -1 \Rightarrow t = a$

$x = 1 \Rightarrow t = b$

$$\int_a^b f(t)\,dt = \int_{-1}^{1} f\left(\frac{b-a}{2} x + \frac{b+a}{2}\right)\left(\frac{b-a}{2}\right) dx = \int_{-1}^{1} g(x)\,dx$$

Ex-1. Use trapezoidal rule with four steps to estimate the rule of integral $\int_0^2 \frac{x}{\sqrt{2+x^2}}\,dx$

Sol. taking $n = 4$

$$h = \frac{x_n - x_0}{n} = \frac{2-0}{4} = 0.5$$

x	0	0.5	1	1.5	2
y	0	0.333	0.57735	0.72760	0.816459
	y_0	y_1	y_2	y_3	y_4

$$\int_0^2 y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots y_{n-1})\right]$$

$$= \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.5}{2}\left[(0 + 0.816459) + (0.333 + 0.57735 + 0.72760)\right]$$

$$= 1.02326$$

Ans

Ex. Evaluate $\int_0^1 \frac{2x}{1+x^2}\,dx$ using Simpson's $\frac{1}{3}$ rule.

Sol taking $n = 4$

$$h = \frac{1-0}{4} = 0.25$$

x	0	0.25	0.50	0.75	1.00
$y = \frac{2x}{1+x^2}$	0	0.47	0.80	0.96	1.00
	y_0	y_1	y_2	y_3	y_4

$$\int_0^1 \frac{2x}{1+x^2}\,dx = \frac{h}{3}\left[(y_0+y_4) + 2y_2 + 4(y_1+y_3)\right]$$

$$= \frac{0.25}{3}\left[(0+1) + 2\times 0.8 + 4(0.47+0.96)\right]$$

$$= 0.69 \quad \underline{\text{Ans}}.$$

Ex.

Time(m)	2	4	6	8	10	12
Velocity (km/h)	22	30	27	18	7	0

Apply Simpson's $\frac{3}{8}$ rule to find the distance covered by the car

Sol. The velocity of the car is given in km/hr. Let us Convert the velocity to km/min.

Time(m)	2	4	6	8	10	12
Velocity (km/min)	$\frac{22}{60}$	$\frac{30}{60}$	$\frac{27}{60}$	$\frac{18}{60}$	$\frac{7}{60}$	$\frac{0}{60}$
	y_0	y_1	y_2	y_3	y_4	y_5

$$\int_2^{12} y\,dx = \frac{3h}{8}\left[(y_0+y_5) + 3(y_1+y_2+y_4) + 2(y_3)\right]$$

$$= \frac{3\times 2}{8}\left[\left(\frac{22}{60}+0\right) + 3\left(\frac{30}{60}+\frac{27}{60}+\frac{7}{60}\right) + 2\times\frac{18}{60}\right]$$

$$= 3.125 \text{ KM}$$