M.A./ M.Sc. Semester II-Complex Analysis

This is the remaining part of the Syllabus.

1 Hurwitz's Theorem

Proposition 1 ((Hurwitz's Theorem)) *Let $U = B(0,R)$ and supose $< f_n >$ be a sequence in $H(U)$ converges to f. Let $0 < r < R$ be such that f has no zero on the circle C of radius r at 0. Then there is n_0 such that for $n \geq n_0$, each f_n has the same number of zeros inside C as f does.*

Proof. Since C is compact and $|f(z)| > 0$ on C, there is $\delta > 0$ such that

$|f(z)| \geq \delta > 0$ for z on C. Let n_0 be such that $|f_n(z)| \geq \delta/2$ for all $z \in C$ and $n \geq n_0$. Then for $z \in C$ and for $n \geq n_0$,

$$\left|\frac{1}{f_n(z)} - \frac{1}{f(z)}\right| = \frac{|f_n(z) - f(z)|}{|f_n(z)|\,|f(z)|} \leq \frac{2}{\delta^2}\,|f_n(z) - f(z)|$$

which proves that $< 1/f_n >$ converges uniformly to $1/f$ on C. Further, since $< f'_n >$ converges uniformly to f'on C, we have $< f'_n/f_n >$ converges to f'/f on C. Hence, $\frac{1}{2\pi i}\int_C (f'_n/f_n)$ converges uniformly to $\frac{1}{2\pi i}\int_C (f'/f)$ or

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_C (f'_n/f_n) = \frac{1}{2\pi i}\int_C (f'/f)$$

which by the *Argument Principle* proves the result. ■

2 Residue at the point at infinity

If a is an isolated singularity of f, then there is a circle of radius $r > 0$ such that f is holomorphic on $\{z : 0 < |z - a| < r\}$ and

$$\frac{1}{2\pi i}\int_C f(\zeta)\mathrm{d}\zeta = \mathrm{Res}(f, a).$$

Keeping this in mind, if ∞ is an isolated singularity of f, we define

$$\mathrm{Res}(f, \infty) = -\frac{1}{2\pi i}\int_C f(\zeta)\mathrm{d}\zeta, \tag{1}$$

where f is holomorphic outside $B(0,R)$ except ∞.

Proposition 2 *If f has only finitly many poles $p_1, p_2, ..., p_n$, then*

$$Res(f, \infty) + \sum_{j=1}^{n} Res(f, p_j) = 0.$$

Proof. Let $R > 0$ be such that all the poles are inside the circle $|z| = R$. Then by Residue theorem

$$\frac{1}{2\pi i}\int_C f(\zeta)\mathrm{d}\zeta = \sum_{j=1}^{n}\mathrm{Res}(f, p_j)$$

which by (1) proves the result. ∎

Proposition 3 *If f has an isolated singularity at ∞, then $Res(f,\infty) = -Res(g, 0)$, where*

$$g(z) = \left(1/z^2\right) f(1/z).$$

Proof. Let $R > 0$ be such that $\mathrm{Res}(f,\infty) = -\frac{1}{2\pi i}\int_C f(\zeta)\mathrm{d}\zeta$, where C is the circle $|z| = R$ and f has no singularity outside C except ∞. If we take $\zeta = 1/w$, then $\mathrm{d}\zeta = -1/w^2\mathrm{d}w$ and the circle C is tranformed by this inversion to the circle $C_1 : |w| = 1/R$ oriented negatively. Hence,

$$\begin{aligned} Res(f,\infty) &= -\frac{1}{2\pi i}\int_C f(\zeta)\mathrm{d}\zeta \\ &= -\frac{1}{2\pi i}\int_{C_1}\left(1/w^2\right) f(1/w)\mathrm{d}w \\ &= -\frac{1}{2\pi i}\int_{C_1} g(w)\mathrm{d}w, \end{aligned}$$

where g has no singularity inside C_1 except 0.Thus $\frac{1}{2\pi i}\int_{C_1} g(w)\mathrm{d}w$ =Res$(g, 0)$ which proves the result. ∎

With the use of the results proved in the Propositions 2 and 3, we may find integrals of the functions having large number of poles.

Example 1 *Evaluate $I = \int_C \frac{dz}{(z-5)(z^{17}-1)}$, where C is the circle of radius 2 at the origin.*

Let $f(z) = \frac{1}{(z-5)(z^{17}-1)}$. Then by Residue theorem $I = 2\pi i\sum_{j=1}^{17}\mathrm{Res}(f, p_j)$, where $p_j's$ are 17 , 17th roots of unity. Obviously this sum is not easy to compute but in view of the abve Propositions, we have

$$\mathrm{Res}(f,\infty) + \sum_{j=1}^{17}\mathrm{Res}(f, p_j) + \mathrm{Res}(f, 5) = 0$$

and

$$Res(f,\infty) = -Res(g, 0),$$

where

$$g(z) = \frac{z^{16}}{(1-5z)\left(1-z^{17}\right)}$$

and

$$Res(g, 0) = 0.$$

Thus

$$\sum_{j=1}^{17} \text{Res}(f, p_j) = -\text{Res}(f, 5) = -\frac{1}{5^{17} - 1}$$

and

$$I = -\frac{2\pi i}{5^{17} - 1}.$$

3 Analytic Continuation

Let f_1 and f_2 be two functions analytic, respectively, analytic in the domains D_1 and D_2 and let in the region $D_1 \cap D_2$, $f_1(z) = f_2(z)$, then f_1 and f_2 are called the analytic continuation of each other from one domain to another.

For example: Let $f_1(z) = \sum_{n=0}^{\infty} z^n$ $(|z| < 1)$ and $f_2(z) = \frac{1}{1-z}$ $(z \neq 1)$. Then the function f_2 is an analytic continuation of $f_1(z)$.

Analytic continuation is a property of analytic functions. Using this property, we have following results:

Theorem 4 *If $f(z)$ is analytic in a domain D and let $f(z) = 0$ at some point or at some part in D, then $f(z) = 0$ throughout D.*

Proof. Let z_0 be a point in D such that $f(z_0) = 0$. Then in a Taylor's series of $f(z)$ in some nbh. N_0 $(\subset D)$ of z_0 coefficients $a_n = \frac{f^{(n)}(z_0)}{n!} = 0$ $\forall n$ and hence, $f(z) = 0$ at each poit of N_0. In this way, we may see that $f(z) = 0$ throughout D. ■

Theorem 5 *There can not be more that one continuations in the same domain.*

Proof. Let $f(z)$ be analytic in D and f_1 and f_2 be two analytic continuations of f in the same domain D_1. That is we have $f = f_1$ in $D \cap D_1$ and also $f = f_2$ in $D \cap D_1$ which implies that $f_1 = f_2$ in $D \cap D_1$, where $D \cap D_1$ is a part of D_1. Thus by (i), $f_1 = f_2$ throughout D_1. This proves the uniqueness property on analytic continuation. ■

The best method of analytic continuation is known as the *power series method of analytic continuation* which is described as follows:

Let

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^n \tag{2}$$

be the Taylor's series of the function $f_1(z)$ at z_1. Then it is convergent within the circle of radius $r_1 = \lim_{n\to\infty} |a_n|^{1/n}$. Let $D_1 := (z : |z - z_1| < r_1)$. Then f_1 is analytic in D_1. Let L be a curve joining z_1 to another point z_n outside D_1.

We perform analytic continuation of f_1 from D_1 to $D_n := (z : |z - z_n| < r_r)$ as follows: Let z_2 be any point on L lying within D_1. Then from (2), we may find

$$f_1^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z - z_1)^{n-k} = \sum_{m=0}^{\infty} \frac{(m+k)!}{m!} a_{m+k} (z - z_1)^m$$

and hence,

$$\frac{f_1^{(n)}(z_2)}{n!} = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!n!} a_{m+n} (z_2 - z_1)^m =: b_n.$$

Thus a Taylor's series of the function $f_1(z)$ at z_2 is given by

$$\sum_{n=0}^{\infty} b_n (z - z_2)^n$$

which converges to $f_2(z)$ (say) within the circle of radius $r_2 = \lim_{n\to\infty} |b_n|^{1/n}$. Let $D_2 := (z : |z - z_2| < r_2)$. Then f_2 is an analytic continuation of f_1 from D_1 to D_2. Clearly, $f_1 = f_2$ in the common region $D_1 \cap D_2$. Continuing in this way, we can get a Taylor's series of the function $f_1(z)$ at z_n which converges to the function f_n within the disc D_n. The function f_n is an analytic continuation of f_1 from D_1 to D_n along the curve L.

Some times the continuation of a power series is not possible beyond its region of convergence through any small arc of its circle of convergence, in that case the circle of convergence is called a *natural boundary*.

Example 2 *The circle of convergence of the power series*

$$f(z) = 1 + z + z^2 + z^4 + z^8 + ... = 1 + \sum_{n=0}^{\infty} z^{2^n}$$

is a natural boundary.

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