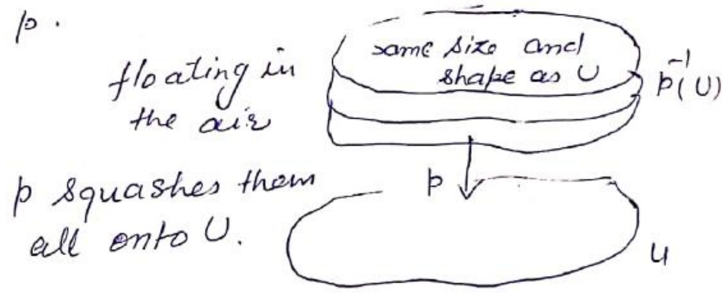
Covering Spaces.

Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be evenly covered by p if the inverse image of $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E. Such that $\forall\ \alpha$, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called the partition of $p^{-1}(U)$ into slices.

ie if $p : E \to B$, $p^{-1}(U) = \bigcup \{ V_i : i = 1, 2, \dots, V_i \in E \}$

and $\forall \alpha$, $p/V_\alpha : V_\alpha \longrightarrow U$ is a homeomorphism.

Now if U is evenly covered by p and $W \subseteq U$ (W is an open set) then W is also evenly covered by p.



Def Let $p : E \to B$ be continuous and surjective. If every point $b \in B$ has a nbd of U that is everenly covered by p, then p is called a covering map and E is called the covering space of B.

Now if $p : E \to B$ is a covering map then $\forall\ b \in B$, the subspace $p^{-1}(b) \in E$ has a discrete topology.

For, each slice V_α is open in E and $V_\alpha \cap p^{-1}(b)$ at a single point, therefore the point is open in $p^{-1}(b)$

~~Prop~~ ex Let $S^1 = \{(x,y) \in R^2 : x^2 + y^2 = 1\}$. The map $p : R \to S^1$ given by the equation $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map.

Proposition : If $p : E \to B$ is a covering map then p is open

Proof - For suppose that,

if A is an open set of E. Given $x \in p(A)$, choose a nbd U of x. ie evenly covered by p.

Let $\{V_\alpha\}$ be the partition of $p^{-1}(U)$ into slices.

Then for any $y \in A$, $p(y) = x$.

Let V_β be the slice containing y.

The set $V_\beta \cap A$ is open in E. and hence open in V_β.

because p maps V_β homeomorphically onto U.

ie the set $p(V_\beta \cap A)$ is open in U.

$\Rightarrow$ open in B. $\Rightarrow$ a nbd of x contained in $P(A)$.

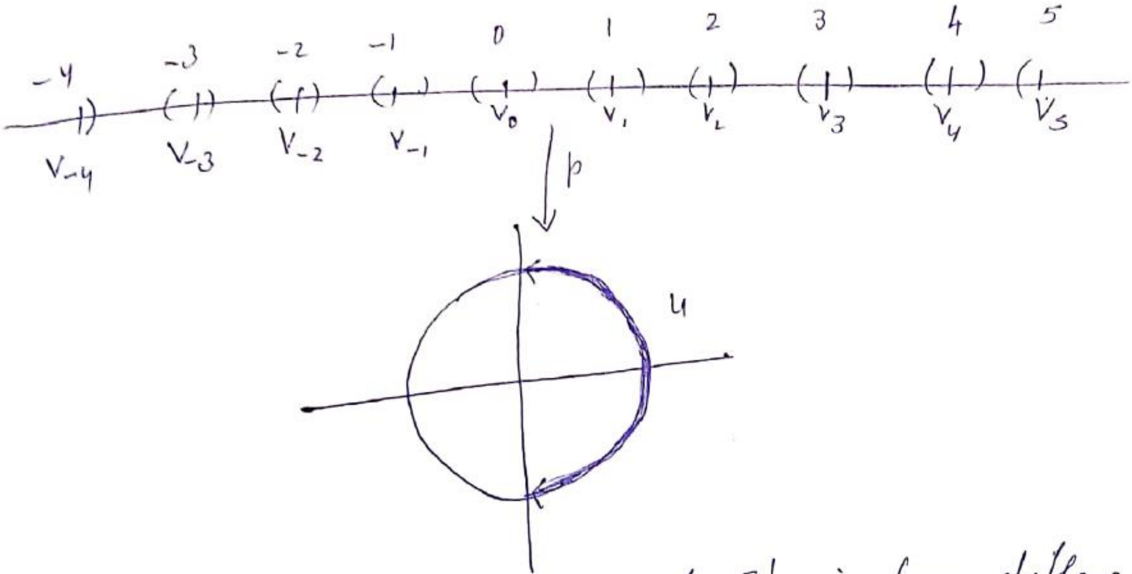
$\Rightarrow$ p is open.

Theorem. The map $p : R \to S^1$ given by the equation

$$p(x) = (\cos 2\pi x,\ \sin 2\pi x)$$

is a covering map

Proof. One can picture p as a function that wraps the real line R around the circle S^1 and in the process maps each interval $[n, n+1]$ onto S^1.



Let us define the equation of circle S^1 in four different ways $\{(x,y) \in R^2 : x^2+y^2 = 1,\ x \geq 0\}$

Right half plane $\quad {}_rS^1 = \{(x,y) \in R^2 : x^2+y^2 = 1,\ x \leq 0\}$

left half plane $\quad {}_LS^1 = \{(x,y) \in R^2 : x^2+y^2 = 1,\ y \geq 0\}$

upper half plane $\quad {}_uS^1 = \{(x,y) \in R^2 : x^2+y^2 = 1,\ y \leq 0\}$.

Lower half plane $\quad {}_{lw}S^1 = $

Consider the subset U of S^1 consisting of those points having positive first coordinate (Right half plane). The set $p^{-1}(U)$ consists of those points x for which $\cos 2\pi x$ is positive. i.e. $V_n = \left(n - \frac{1}{4},\ n + \frac{1}{4}\right)\ \forall\ n \in I$.

It is the union of intervals, V_n. Now restricted to any closed interval $\bar{V}_n$, the map p is injective because $\sin 2\pi x$ is strictly monotonic on such interval.

Now p carries $\bar{V}_n$ surjectively onto $\bar{U}$ and V_n to U, by intermediate theorem.

Intermediate Value Theorem - Let $f : X \to Y$ be a continuous map. If a & b are two pts in X and $r \in Y$ and $f(a) \le r \le f(b)$ then $\exists$ a pt $c \in X$ such that $f(c) = r$.

Now in this case we take X as a closed interval in R and Y to be R.

ie $f : [\ \ ] \to R$

$\because$ the closed interval on the real line is compact.

Since $\bar{V}_n$ is compact, $p|\bar{V}_n$ is a homeomorphism of $\bar{V}_n$ with $\bar{U}$

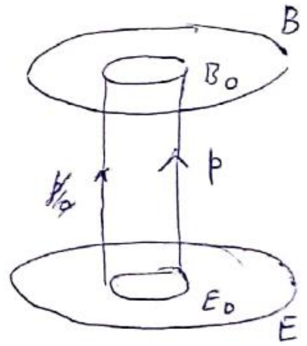
ie $p|V_n$ is a homeomorphism of V_n with U.

The similar arguments can be applied to the intersections of S^1 with the S^1_U, $S^1_{L_w}$, S^1_L planes.

These open sets covers S^1, and each of them is evenly covered by p.

Therefore p is a covering map.

<u>Theorem</u> - Let $p : E \to B$ be a covering map. If B_0 is a sub space of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map.



Proof. Given $b_0 \in B_0$, let U be the open set in B containing b_0 that is evenly covered by p.

Let $\{V_\alpha\}$ be the partition of $p^{-1}(U)$ into slices.

Then $U \cap B_0$ is a nbd of b_0 in B_0.

and the set $V_\alpha \cap E_0$ are disjoint open sets in E_0

whose union is $p^{-1}(U \cap B_0)$

ie $p^{-1}(U \cap B_0) = \cup\{(V_\alpha \cap E_0) : E_0 \in E\}$.

ie $p^{-1}(U) \cap p^{-1}(B_0) = \cup\{(V_\alpha \cap E_0) : E_0 \subseteq E\}$.

Each is mapped ~~onto~~ homeomorphically onto $U \cap B_0$ by p

Hence the map $p_0 : E_0 \to B_0$ is restricting by p is a covering map.

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Theorem. If $p : E \to B$ and $p' : E' \to B'$ are covering maps, then $p \times p' : E \times E' \longrightarrow B \times B'$ is a covering map.

Proof. Let $b \in B$ and $b' \in B'$. Let U and U' be the nbds of b and b' respectively, that are evenly covered by p and p'.
Let $\{V_\alpha\}$ and $\{V'_\beta\}$ be the partitions of $p^{-1}(U)$ and $(p')^{-1}(U')$ into slices.
Then the inverse images under $p \times p'$ of open sets $U \times U'$ is the union of all the sets $V_\alpha \times V'_\beta$.

ie $(p \times p')^{-1}(U \times U') = \cup\{V_\alpha \times V'_\beta\}$.

ie there are disjoint open sets of $E \times E'$ and each is mapped homeomorphically onto $U \times U'$ by $p \times p'$.

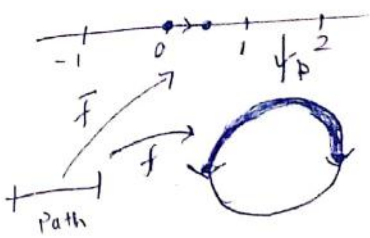
ex – The space $S^1 \times S^1 = T$ is called Torus.
The map $p \times p : R \times R \longrightarrow S^1 \times S^1$ is a covering map of the torus by the plane R^2.

Def Let $p : E \to B$ be a map. If f is a continuous map of some space X into B, a lifting of f is a map $\tilde{f} : X \to E$ s.t. $p \circ \tilde{f} = f$

$$\boxed{p \circ \tilde{f} = f}$$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

ex Let $p : R \to S^1$ is a covering map.



The path $f : [0,1] \to S^1$ begining at $b_0 = (0,1)$ given by $f(s) = (\cos \pi s, \sin \pi s)$. Let $\tilde{f}(s) = s/2$ be the path. Then f lifts $\tilde{f}$ begining at 0 and ending at 1.

Now $(p \circ \tilde{f})(s) = p(\tilde{f}(s)) = p\left(\frac{s}{2}\right) = \left(\cos 2\pi \cdot \frac{s}{2}, \sin 2\pi \cdot \frac{s}{2}\right) = f(s)$

ie $p \circ \tilde{f} = f$. where $p(s) = (\cos 2\pi s, \sin 2\pi s)$

Ex If $g(s) = (\cos \pi s, -\sin \pi s)$, $p(s) = (\cos 2\pi s, \sin 2\pi s)$ is a cov. map

~~then~~ and path $\bar{g}(s) = -\frac{s}{2}$.

Then $(p \circ \bar{g})(s) = p(\bar{g}(s)) = p(-\frac{s}{2}) = \left(\cos\left(2\pi \cdot \left(-\frac{s}{2}\right)\right), \sin 2\pi\left(-\frac{s}{2}\right)\right)$
$= (\cos \pi s, -\sin \pi s) = g(s)$

$\Rightarrow p \circ \bar{g} \simeq g$.

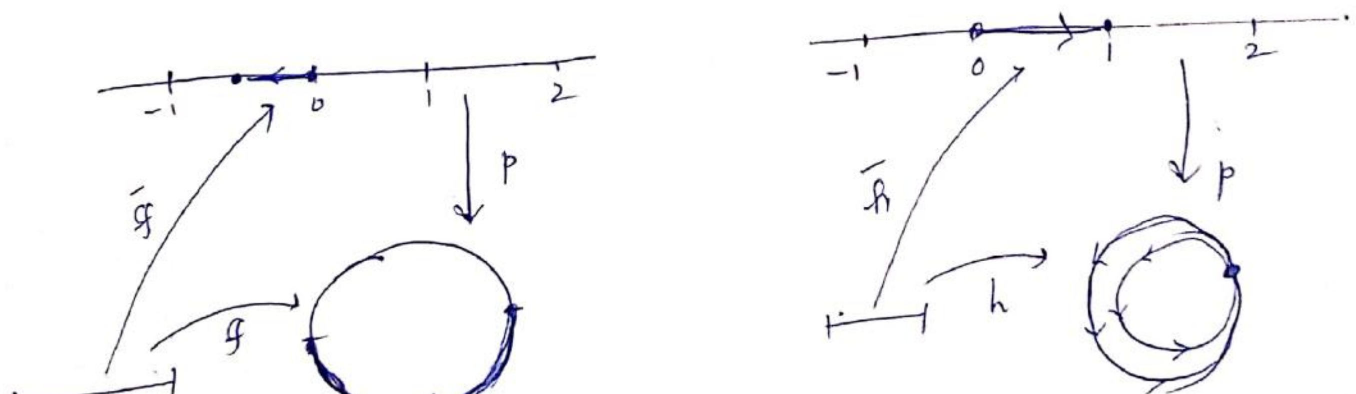
Hence the path $g(s)$ lifts to the path $\bar{g}(s)$ begining at 0 and ending at $-1/2$.

Ex If $h(s) = (\cos 4\pi s, \sin 4\pi s)$ be a path and path $\bar{h}(s) = 2s$, $p(s) = (\cos 2\pi s, \sin 2\pi s)$ is a covering map.

Then $(p \circ \bar{h})(s) = p(\bar{h}(s)) = p(2s)$
$= (\cos 4\pi s, \sin 4\pi s) = h(s)$

$\Rightarrow p \circ \bar{h} = h$.

Hence $h(s)$ lift to the path $\bar{h}(s)$ begining at 0 and ending at 2. Actually h wraps the interval $[0,1]$ around the circle twice and ends at the number 2.



Lemma 1: Let $p: E \to B$ be a covering map, let $p(e_0) = b_0$. Any path begining at b_0 defined as $f: [0,1] \to B$, has a unique lifting of a path $\bar{f}$ in E beginning at e_0

Lemma 2: Let $p: E \to B$ be a covering map and let $p(e_0) = b_0$. Let the map $F: I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map

$$\tilde{F}: I \times I \to E$$

Such that $\bar{F}(0,0) = e_0$. If F is a path homotopy then $\bar{F}$ is a path homotopy.

Theorem: Let $p: E \to B$ be a covering map. Let $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1; let $\bar{f}$ and $\bar{g}$ be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic, then $\bar{f}$ and $\bar{g}$ end at the same point of E and are path homotopic

Proof —

Let $F: I \times I \to B$ be the path homotopy between f and g. Then $F(0,0) = b_0$. Let $\bar{F}: I \times I \to E$ be the lifting of F to E such that $\bar{F}(0,0) = e_0$. Then $\bar{F}$ is a path homotopy by above lemma 2. So that

$$\bar{F}(0 \times I) = \{e_0\} \text{ and } \bar{F}(1 \times I) = \{e_1\} \text{ one point set.}$$

The restriction $\bar{F}|I \times 0$ of $\bar{F}$ to the bottom edge of $I \times I$ is a path on E beginning at e_0 that is a lifting of $F|(I \times 0)$. By uniqueness of path liftings, $\bar{F}(s,0) = \bar{f}(s)$. Similarly $\bar{F}|I \times 1$ is a path on E is a lifting of $F|(I \times 1)$, and it begins at e_0 because $\bar{F}(0 \times I) = \{e_0\}$. By uniqueness of path liftings $\bar{F}(s,1) = \bar{g}(s)$. Hence both $\bar{f}$ and $\bar{g}$ end at e_1 and $\bar{F}$ is a path homotopy between them.

Def. Let $p : E \to B$ be a covering map and let $p(e_0) = b_0$ $\forall b_0 \in B$.
if $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E
that begins at e_0. Let $\varphi([f])$ denote the end point $\tilde{f}(1)$
of $\tilde{f}$. Then φ is well defined map

$$\varphi : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

φ is called lifting correspondence derived for the covering map p. It depends the choice of the point e_0.

Theorem - Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$.

1. If E is the path connected then lifting correspondence

$$\varphi : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

is onto (surjective).

2. If E is simply connected then it is bijective (one-one onto)

Proof. If E is path connected. then given $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ in E from e_0 to e_1. ie $\tilde{f} : [0,1] \to E$

Then $f = p \circ \tilde{f}$ is a loop in B, at b_0.

Also $\tilde{f}$ is a lifting of f to a path in E beginning at 0.

Hence $\varphi([f]) = e_1$ by the defn

Hence φ is onto.

(ii) Suppose that E is simply connected. and $\varphi[f] = \varphi[g] = e_1$
where $[f], [g] \in \pi_1(B, b_0)$.

Let $\tilde{f}$ and $\tilde{g}$ are the liftings of f and g respectively to the path in E beginning at e_0. Then $\tilde{f}(1) = \tilde{g}(1)$.

Theorem. The fundamental group of the circle is infinite cyclic.

Let $p : R \to S^1$ be a covering map s.t. $p(s) = (\cos 2\pi s, \sin 2\pi s)$. Let $b_0 = p(e_0)$. Then $b_1 = (0,1) = p(e_1)$ ie $p^{-1}(b_0) \in \mathbb{Z}$

Proof Let b_0 be a point ~~of~~ $(1, 0)$ of S^1. We shall construct an isomorphism of the additive group of integers group $\pi_1(S^1, b_0)$ with the group $(\mathbb{Z}, +)$ of integers

Let $p : R \to S^1$ be a covering map given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$

If f is a loop on S^1 based at b_0, let $\tilde{f}$ be the lifting of f to a path on R begining at 0. The point $\tilde{f}(1)$ must be a point of the set $p^{-1}(b_0)$ ie $\tilde{f}(1)$ must be equal to some integer say n. ie the integer depends only on the path homotopy class of f. So that we define the lifting correspondence

$$\varphi : \pi_1(S^1, b_0) \to \mathbb{Z}$$

by letting $\varphi([f])$ be this integer.

$\because$ R is path connected $\Rightarrow$ R is simply connected.

To show that φ is a group isomorphism.

1. The map is onto - Let $n \in p^{-1}(b_0)$ Because R is path connected. Choose a path $\tilde{f} : [0,1] \to R$ in R from 0 to n. Define $f = p \circ \tilde{f}$. Then f is a loop based b_0 in S^1 and $\tilde{f}$ is its lifting to a path in R begining at 0. then $\varphi([f]) = n$.

2. Map is one-one - Assume that $\varphi([f]) = n = \varphi([g])$, to show that $[f] = [g]$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g to the path on R begining at 0; both $\tilde{f}, \tilde{g}$ end at n by hypothesis. Because R is simply connected $\tilde{f}, \tilde{g}$ are path homotopic. Let $\tilde{F}$ be the path homotopy between them. Then $F = p \circ \tilde{F}$ will be path homotopy between f & g.

3. Map φ is homomorphism - Let f and g be two loops in S^1 based at b_0. Let $\tilde{f}, \tilde{g}$ be their liftings to the path on R begining at 0.

Let $\bar{f}(1) = n$ and $\bar{g}(1) = m$. Define a path h on R by the eq.

$$h(s) = \begin{cases} \bar{f}(2s) & 0 \le s \le 1/2 \\ n + \bar{g}(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

h is a path on R begining at 0. Claim that h is a lifting of $f * g$.

$\because$ for all x, $p(n+x) = p(x)$ because the function sine and cosine have period 2π. (Periodic function). Then

$$p(h(s)) = \begin{cases} p(\bar{f}(2s)) & 0 \le s \le 1/2 \\ p(n + \bar{g}(2s-1)) & 1/2 \le s \le 1 \end{cases}$$

$$= \begin{cases} (p \circ \bar{f})(2s) \\ p(n) + (p \circ \bar{g})(2s-1) \end{cases}$$

$$= \begin{cases} f(2s) \\ g(2s-1) \end{cases} \Rightarrow f * g \cdot (s)$$

$\Rightarrow$ $p \circ h = f * g$.

$\Rightarrow$ h is a lifting of $f * g$. which begins at zero.

By def $\varphi([f * g])$ is $h(1)$ which equals to $n + m$. Therefore

$$\varphi([f * g]) = \varphi([f]) + \varphi([g])$$

Hence homomorphism.

The fundamental theorem of algebra -

Theorem - A polynomial equation

$$x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 = 0$$

of degree $n > 0$ with real or complex coefficients has at least one (real or complex) root.

Proof. Consider the map $f : S^1 \to S^1$ given by $f(z) = z^n$, where z is a complex number. We show that the induced homomorphism f_* of fundamental groups is injective.

Let $p_0 : I \to S^1$ be the ~~standard~~ standard loop in S^1.

$$p_0(s) = (\cos 2\pi n s, \sin 2\pi n s) = (e^{2\pi i})^n$$

This loop lifts to the path $s \to ns$ in the covering space R. Therefore, the loop $f \circ p_0$ corresponds to the integer n under the standard isomorphism of $\pi_1(S^1, b_0)$ with the integers, where p_0 corresponds to the number 1. Thus "f_* is multiplication by n" in the fundamental group of S^1. Hence f_* is one-one.

Now to show that if $g : S^1 \to R^2 - 0$ is the map $g(z) = z^n$ then g is not null homotopic.

The map g ~~is~~ equals a map f followed by the inclusion map $j : S^1 \to R^2 - 0$. Now f_* is injective ~~therefore~~ and j_* is injective because S^1 is the retract of $R^2 - 0$.

$\Rightarrow g_* = j_* \circ f_*$ is injective. Thus g is not null homotopic.

Now to prove a special case of the theorem - Given a polynomial equation $x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$.

Assume that.

$$|a_{n-1}| \ast |a_{n-2}| + \cdots |a_1| + |a_0| < 1.$$

and to show that the equation has a root lying in the unit ball B^2.

Assume that it has no such root.

Now define a map $k : B^2 \to R^2 - 0$ by the equation

$$k(z) = z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0 = 0$$

Let h be the restriction of k to S^1. Becaush extends to a map of unit ball into $R^2 - 0$, the map h is null homotopic.

On the other hand, define a homotopy F between the maps f and g as above defined ($g_* = j_* \circ f_*$), since g is not a null homotopic then we get a contradiction.

Define $F : S^1 \times I \to R^2 - 0$ by the equation

$$F(z,t) = z^n + t(a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0)$$

$F(z,t) \neq 0 \quad (e_{x_0})$ because

$$|F(z,t)| \geq |z^n| - |t(a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)|$$

$$\geq 1 - \cancel{t} \, t(|a_{n-1}| z^{n-1} + \cdots + |a_1| z + |a_0|)$$

$$= 1 - t(|a_{n-1}| + \cdots |a_1| + |a_0|) \geq 0$$

Now to prove the general case. Given a polynomial equation

$$x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Let us choose a real number $c > 0$ and substitute $x = cy$

To obtain the equation

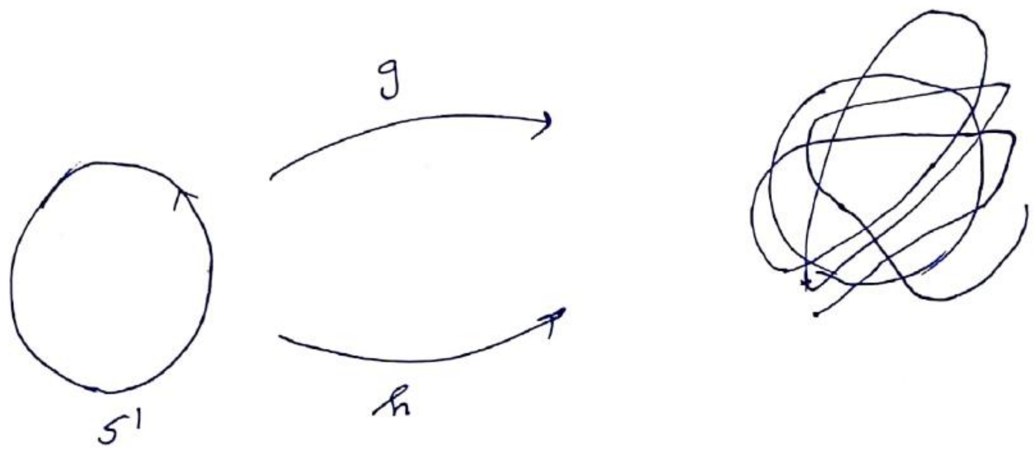
$$(cy)^n + a_{n-1}(cy)^{n-1} + \cdots + a_1(cy) + a_0 = 0$$

or $y^n + \frac{a_{n-1}}{c} y^{n-1} + \cdots + \frac{a_1}{c^{n-1}} y + \frac{a_0}{c^n} = 0$

If this equation has the root $y = y_0$ then the orignal equation has the root $x_0 = c y_0$. So we need merely choose c large enough that

$$\left|\frac{a_{n-1}}{c}\right| + \left|\frac{a_{n-2}}{c^2}\right| + \cdots + \left|\frac{a_1}{c^{n-1}}\right| + \left|\frac{a_0}{c^n}\right| < 1$$

to reduce the theorem to the special case as considered above.



using the theorem —.

Let $h : S^1 \to X$ be a continuous map. Then following are equi

(i) h is null homotopic

(ii) h extends a continuous map $k : B^2 \to X$

(iii) h_* is a trivial homomorphism of fundamental group

Def If $A \subset X$, a retraction of X onto A is a conti map $r : X \to A$ such that $r|A$ is the identity map of A. If such a map r exists A is retract of X.