Circuit Theory

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CIRCUIT THEORY is an important and perhaps the oldest branch of electrical engineering. A circuit is an interconnection of electrical elements: passive elements such as resistances, capacitances, inductances, active elements, and sources (or excitations). Two variables—namely, voltage and current variables—are associated with each circuit element. There are two aspects to circuit theory: analysis and design. In circuit analysis, we are interested in determination of the values of currents and voltages in different elements of the circuit, given the values of the sources or excitations. On the other hand, in circuit design, we are interested in the design of circuits, which exhibit certain prespecified voltage or current charactersitics at one or more parts of the circuit. In this chapter, we will confine our discussion to certain aspects of circuit analysis.

The behavior or dynamics of a circuit is described by three systems of equations determined by Ohm's law, Kirchhoff's voltage law, and Kirchhoff's current law. Ohm's law specifies the relationship between the voltage and current variables associated with a circuit element. This relationship is not specified for independent sources. Also, this relationship could be linear or nonlinear. If the relationship is linear, then the circuit element is called a *linear element*; otherwise, it is called a *nonlinear element*. A circuit is linear if it contains only linear elements besides independent sources. Kirchhoff's voltage specifies the dependence among the voltage variables in the circuit, and Kirchhoff's current law specifies the dependence among the current variables in the circuit.

The systems of equations determined by the application of Kirchhoff's voltage and current laws depend on the structure or the graph of the circuit. In other words, they depend only on the way the circuit elements are interconnected. Thus, the graph of a circuit plays a fundamental role in the study of circuits. Several interesting properties of circuits depend only on the structure of the circuits. Thus, the theory of graphs has played a fundamental role in discovering structural properties of electrical circuits.

In this chapter we shall develop most of those results that form the foundation of graph theoretic study of electrical circuits. A comprehensive treatment of these developments may be found in Swamy and Thulasiraman (1981). All theorems in this chapter are stated without proofs. Our discussion here follows closely our development in the Graph Theoretic Foundation of Circuit Analysis chapter in Chen (2001).

I. GRAPH THEORY: BASIC CONCEPTS AND RESULTS

Our development of graph theory is self-contained, except for the definitions of standard and elementary results from set theory and matrix theory.

Encyclopedia of Physical Science and Technology, Third Edition, Volume 2

• **Graph:** A graph $G = (V, E)$ consists of two sets, a finite set $V = (v1, v_2, \ldots, v_n)$ of elements called *vertices* and a finite set $E = (e_1, e_2, \ldots, e_n)$ of elements called *edges*.

• **Directed and undirected graph:** If the edges of G are identified with ordered pairs of vertices, then G is called a directed or an *oriented graph*; otherwise,it is called an *undirected* or an *unoriented graph.*

Graphs permit easy pictorial representations. In a pictorial representation each vertex is represented by a dot and each edge is represented by a line joining the dots associated with the edge. In directed graphs, an orientation or direction is assigned to each edge. If the edge is associated with the ordered pair (v_i, v_j), then this edge is oriented from v_i to v_j. If an edge e connects vertices v_i and v_j, then it is denoted by $e = (v_i, v_j)$. In a directed graph, (v_i, v_j) refers to an edge directed from (v_i, v_j). A graph and a directed graph are shown in Fig. 1. Unless explicitly stated, the term "graph" may refer to a directed graph or an undirected graph.

• **End vertices:** The vertices v_i and v_j associated with an edge are called the *end vertices* of the edge.

• **Parallel edges:** All edges having the same pair of end vertices are called *parallel* edges. In a directed graph parallel edges refer to edges connecting the same pair of verices v_i and v_j the same way from v_i to v_j or from v_j to v_i . For instance, in the graph of Fig. 1a, the edges connecting v_1 and v_2 are parallel edges. In the directed graph of Fig. 1b the edges connecting v_3 and v_4 are parallel edges. However, the edges connecting v_1 and v_2 are not parallel edges because they are not oriented in the same way.

• **Self loop:** If the end vertices of an edge are not distinct, then the edge is called a *self loop*. The graph of Fig. 1a has one self loop and the graph of Fig. 1b has two self loops.

An edge is said to be *incident on* its end vertices. In a directed graph the edge (v_i, v_j) is said to be *incident out* of v_i and is said to be *incident into* v_j. Vertices v_i and v_j are adjacent if an edge connects v_i and v_j.

• **Degee:** The number of edges incident on a vertex v_i is called the *degree* of v_i and is denoted by $d(v_i)$.

• **In-degree:** In a directed graph, $d_{in}(v_i)$ refers to the number of edges incident into vertex v_i, and it is called the *in-degree*.

• **Out-degree:** In a directed graph, $d_{out}(v_i)$ refers to the number of edges incident out of the vertex v_i.

• **Isolated vertex:** If $d(v_i) = 0$, then v_i is said to be an *isolated vertex*.

• **Pendant vertex:** If $d(v_i) = 1$, then v_i is said to be a *pendant vertex*.

A self loop at a vertex v_i is counted twice while computing $d(v_i)$. As an example,in the graph of Fig. 1a, $d(v_1) = 3$, $d(v_4) = 3$, and v_5 is an isolated vertex. In the directed graph of Fig. 1b, $d_{in}(v_1) = 3$, and $d_{out}(v_1) = 2$.

Note that in a directed graph, for every vertex v_i,

$$d(v_i) = d_{in}(v_i) + d_{out}(v_i)$$

Theorem 1

1. The sum of the degrees of the vertices of a graph G is equal to $2m$, where m is the number of edges of G.
2. In a directed graph with m edges, the sum of the in-degrees and the sum of the out-degrees are both equal to m.

The following theorem is known to be the first major result in graph theory.

Theorem 2

The number of vertices of odd degree in any graph is even.

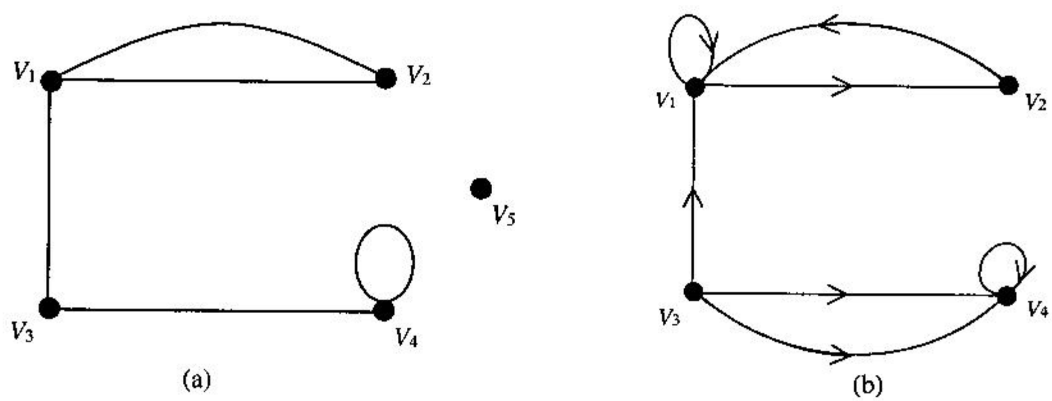

FIGURE 1 (a) An undirected graph, (b) a directed graph.

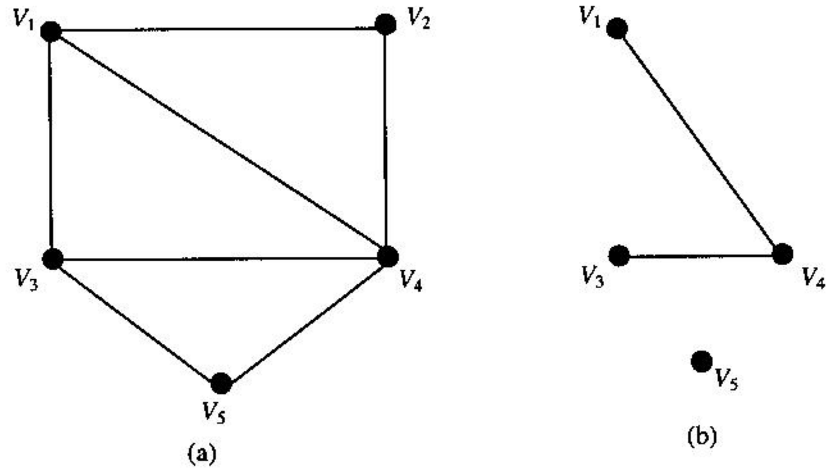

FIGURE 2 (a) Graph G; (b) subgraph G.

Consider a graph $G = (V', E')$. The graph $G' = (V', E')$ is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. As an example, a graph G and a subgraph of G are shown in Fig. 2.

- **Path:** In a graph G a *path P* connecting vertices v_i and v_j is an alternating sequence of vertices and edges starting at v_i and ending at v_j, with all vertices except v_i and v_j being distinct.
- **Directed path:** In a directed graph a path P connecting vertices v_i and v_j is a called a *directed path* from v_i to v_j if all the edges in P are oriented in the same direction as we traverse P from v_i toward v_j.
- **Circuit:** If a path starts and ends at the same vertex, it is called a *circuit*.
- **Directed circuit:** In a directed graph, a circuit in which all the edges are oriented in the same direction is called a *directed circuit*. It is often convenient to represent paths and circuits by the sequence of edges representing them. For example, in the undirected graph of Fig. 3a, P: e_1, e_2, e_3, e_4 is a path connecting v_1 and v_5, and C: $e_1, e_2, e_3, e_4, e_5, e_6$ is a circuit. In the directed graph of Fig. 3b, P: e_1, e_2, e_7, e_5 is a directed path, and C: e_1, e_2, e_7, e_6 is a directed circuit. Note that C: e_7, e_5, e_4, e_1, e_2 is a circuit in this directed graph, although it is not a directed circuit. Similarly, P: e_9, e_6, e_1 is a path but not a directed path.
- **Connected graph:** A graph is *connected* if there is a path between every pair of vertices in the graph; otherwise, the graph is not connected. For example, the graph in Fig. 2a is a connected graph, whereas the graph in Fig. 2b is not a connected graph.
- **Tree:** A *tree* is a graph that is connected and has no circuits.
- **Spanning tree:** Consider a connected graph G, A subgraph of G is a *spanning tree* of G if the subgraph is a tree and contains all the vertices of G. A tree and a spanning tree of the graph of Fig. 4a are shown in Figs. 4b and 4c, respectively.
- **Branches:** The edges of a spanning tree T are called the braches of T.

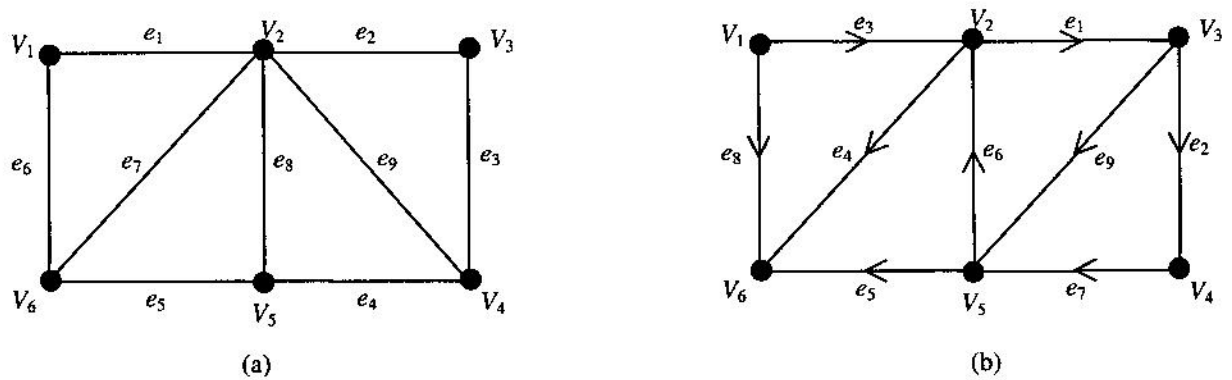

FIGURE 3 (a) An undirected graph; (b) a directed graph.

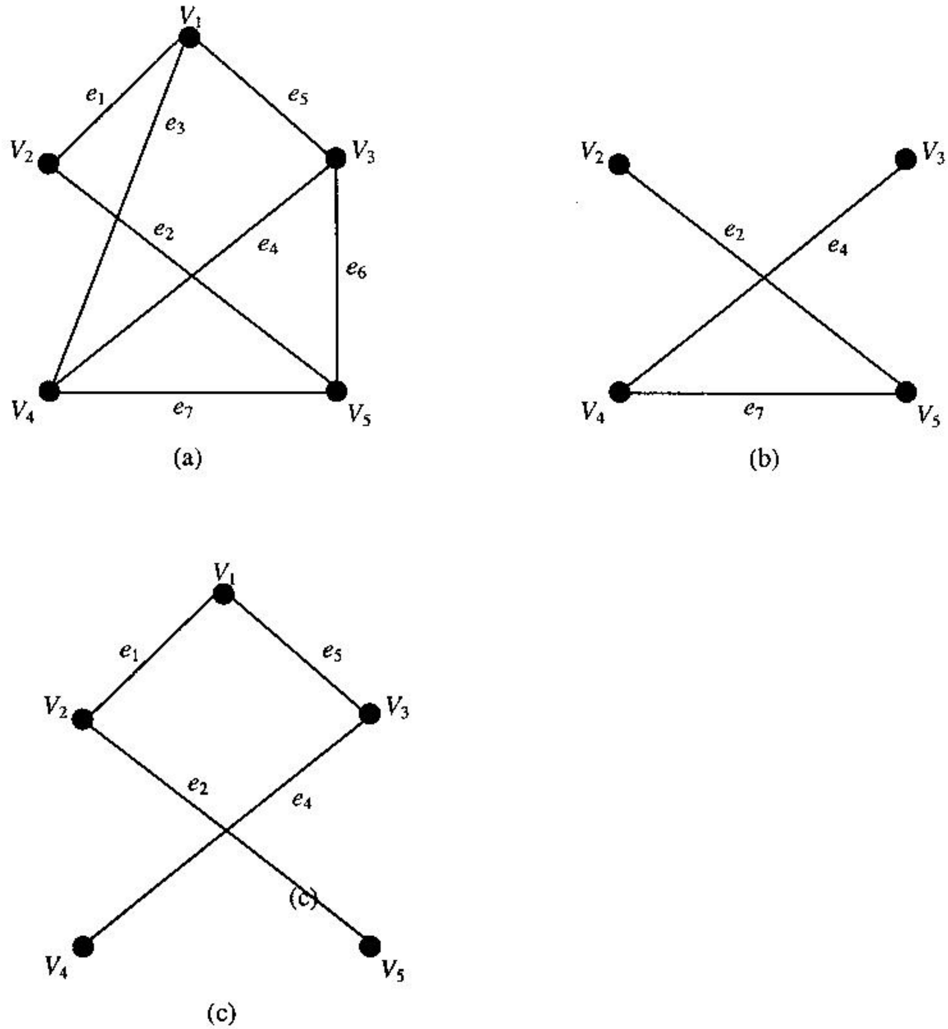

FIGURE 4 (a) Graph G; (b) a tree of graph G; (c) a spanning tree of G.

- **Cospanning tree:** Given a spanning tree of connected graph G, the cospanning tree relative to T is the subgraph of G induced by the edges that are not present in T. For example, the cospanning tree is relative to the spanning tree T if Fig. 4c consists of the edges e_3, e_6, e_7.
- **Chords:** The edges of a cospanning tree are called *chords.*

It can be easily be verified that in a tree exactly one path connects any two vertices. It should be noted that a tree is minimally connected in the sense that removing any edge from the tree will result in a disconnected graph.

Theorem 3

A tree on n vertices has $n-1$ edges.

If a connected graph G has n vertices and m edges, then the rank ρ and nullity μ of G are defined as follows:

$$\rho(G) = n - 1$$
$$\mu(G) = m - n + 1$$

The concepts of rank and nullity have parallels in other branches of mathematics, such as matrix theory.

Clearly, if G is connected, then any spanning tree of G has $\rho = n - 1$ branches and $\mu = m - n + 1$ chords.

A. Cuts, Circuits, and Orthogonality

We introduce here the notions of a cut and a cutset and develop certain results which bring out the dual nature of circuits and cutsets. Consider a connected graph $G = (V, E)$ with n vertices and m edges. Let V_1 and V_2 be two mutually disjoint nonempty subsets of V such that $V = V_1 \cup V_2$; thus, V_2 is the complement of V_1 in V and vice versa. V_1 and V_2 are also said to form a partition of V. Then the set

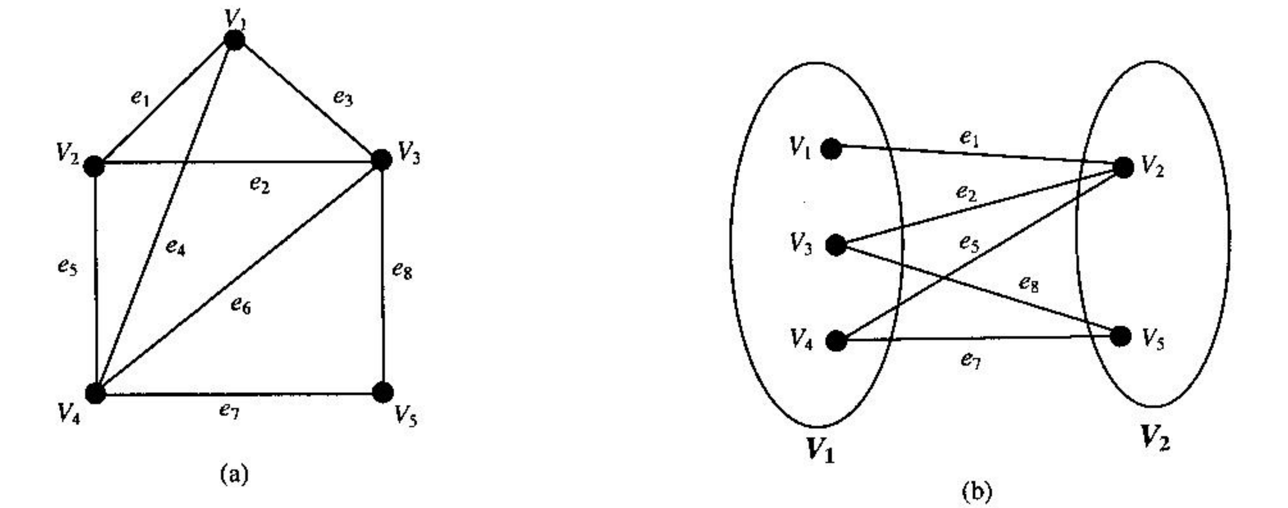

FIGURE 5 (a) Graph G; (b) cut $\langle V_1, V_2\rangle$ of G.

of all those edges which have one end vertex in V_1 and the other in V_2 is called a *cut* of G. As an example, a graph and a cut $\langle V_1, V_2\rangle$ of G are shown in Fig. 5.

The graph G' which results after removing the edges in a cut will not be connected. A *cutset* S of a connected graph G is a minimal set of edges of G such that removal of S disconnects G. Thus, a cutset is also a cut. Note that the minimality property of a cutset implies that no proper subset of a cutset is a cutset.

Consider a spanning tree T of a connected graph G. Let b be a branch of T. Removal of the branch b disconnects T into two trees, T_1 and T_2. Let V_1 and V_2 denote the vertex sets of T_1 and T_2, respectively. Note that V_1 and V_2 together contain all the vertices of G. We can verify that the cut $\langle V_1, V_2\rangle$ is a cutset of G and is called the *fundamental cutset* of G with respect to branch b of T. Thus, for a given graph G and a spanning tree T of G, we can construct $n-1$ fundamental cutsets, one for each branch of T. As an example, for the graph shown in Fig. 5, the fundamental cutsets with respect to the spanning tree $T = [e_1, e_2, \ldots, e_6, e_8]$ are

Branch e_1: (e_1, e_3, e_4)
Branch e_2: (e_2, e_3, e_4, e_5)
Branch e_6: (e_6, e_4, e_5, e_7)
Branch e_8: (e_8, e_7)

Note that the fundamental cutset with respect to branch b contains b. Furthermore, branch b is not present in any other fundamental cutset with respect to T.

Next we identify a special class of circuits of a connected graph G. Again, let T be a spanning tree of G. Because exactly one path exists between any two vertices of T, adding a chord c to T produces a unique circuit. This circuit is called the *fundamental circuit* of G with respect to chord c of T. Note again that the fundamental circuit with respect to chord c contains c, and chord C is not present in any other fundamental circuit with respect to T. As an example, the set of fundamental circuits with respect to the spanning tree $T = (e_1, e_2, e_6, e_8)$ of the graph shown in Fig. 5 is

Chord e_3: (e_3, e_1, e_2)
Chord e_4: (e_4, e_1, e_2, e_6)
Chord e_5: (e_5, e_2, e_6)
Chord e_7: (e_7, e_8, e_6)

B. Incidence, Circuit, and Cut Matrices of a Graph

The incidence, circuit, and cut matrices are coefficient matrices of Kirchoff's equations which describe an electrical network. We next define these matrices and present some properties of these matrices which are useful in the study of electrical networks.

1. Incidence Matrix

Consider a connected directed graph G with n vertices and m edges and having no self loops. The *all-vertex incidence matrix* $A_c = [a_{ij}]$ of G has n rows, one for each vertex, and m columns, one for each edge. The element a_{ij} of A_c is defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } j\text{th edge is incident out of the } i\text{th vertex} \\ -1, & \text{if } j\text{th edge is incident into the } i\text{th vertex} \\ 0, & \text{if the } j\text{th edge is not incident on the } i\text{th vertex} \end{cases}$$

As an example, the A_c matrix of the directed graph in Fig. 6 is given below: