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Thm: Let a_* be a frame with frame constant $B \geq A > 0$ and let $\bar{a}_*$ be the corresponding dual frame. Then

(i) for frames a_* & $\bar{a}_*$

$$x = \sum_{j=1}^{r} \langle x, a_j \rangle \bar{a}_j \quad \forall x \in X.$$

(ii) the image Sy of $y = (y_1, y_2, \cdots y_r) \in Y$ is

$$Sy = \sum_{j=1}^{r} y_j \bar{a}_j.$$

(iii) $\bar{a}_*$ is a frame with frame constants $\frac{1}{A} \geq \frac{1}{B} > 0$.

(iv) The dual frame of $\bar{a}_*$ is a_* i.e.,

$$x = \sum_{j=1}^{r} \langle x, \bar{a}_j \rangle a_j \quad \forall x \in X.$$

Proof:- (i) We can write, $\forall x \in X$,

$$x = G^{-1}G x = G^{-1}(T^* T x).$$

$$= G^{-1}\left(T^*\left(\sum_{j=1}^{r} \langle x, a_j \rangle e_j\right)\right) \qquad (Tx = \sum \langle x, a_j \rangle e_j).$$

$$= G^{-1}\left(\sum_{j=1}^{r} \langle x, a_j \rangle a_j\right) \qquad (T^* e_j = a_j)$$

$$= \sum_{j=1}^{r} \langle x, a_j \rangle \bar{a}_j \qquad (G^{-1} a_j = \bar{a}_j).$$

(ii) We know that $S = G^{-1}T^*$ so, for $y \in Y$,

$$Sy = G^{-1}T^* y = G^{-1}\left(T^*\left(\sum_{j=1}^{r} y_j e_j\right)\right) = G^{-1}\left(\sum_{j=1}^{r} y_j a_j\right)$$

$$= \sum_{j=1}^{r} y_j \bar{a}_j. \qquad (G^{-1} a_j = \bar{a}_j).$$

(iii) Let $\bar{T}$ be the frame operator belonging to the frame $\bar{a}_*$. Since G is self adjoint so, G^{-1} is also self adjoint i.e, $G^{-1} = (G^{-1})^*$. So,

$$(\bar{T}x)_j = \langle x, \bar{a}_j \rangle = \langle x, G^{-1} a_j \rangle = \langle (G^{-1})^* x, a_j \rangle$$

$$= \langle G^{-1}x, a_j \rangle = (T(G^{-1}x))_j \qquad ((Tx)_j = \langle x, a_j \rangle)$$

$$\Rightarrow \quad \bar{T} = TG^{-1}.$$

$$\therefore \|\bar{T}x\|^2 = \|TG^{-1}x\|^2 = \langle TG^{-1}x, TG^{-1}x \rangle$$

$$= \langle T^* T G^{-1}x, G^{-1}x\rangle = \langle G G^{-1}x, G^{-1}x\rangle \quad (T^*T = G)$$
$$= \langle x, G^{-1}x\rangle.$$

Let $(\bar{e}_1, \bar{e}_2, \cdots \bar{e}_n)$ be an orthonormal basis which diagonalize G^{-1}, then

$$\|\bar{T}x\|^2 = \langle x, G^{-1}x\rangle = \sum_i (x)\left(\frac{1}{\lambda_i}\bar{x}_i\right)$$
$$= \begin{cases} \leq \frac{1}{A}\|x\|^2 \\ \geq \frac{1}{B}\|x\|^2. \end{cases}$$

$$\boxed{\therefore \frac{1}{B}\|x\|^2 \leq \|\bar{T}x\|^2 \leq \frac{1}{A}\|x\|^2}$$

$\lambda_i \to$ eigenvalue of G

then $\frac{1}{\lambda_i} \to$ eigenvalue of G^{-1}

$A = \lambda_1 < \lambda_2 \cdots < \lambda_n = B$

$\frac{1}{A} = \frac{1}{\lambda_1} > \frac{1}{\lambda_2} \cdots > \frac{1}{\lambda_n} = \frac{1}{B}$

(IV) The Gram operator for $\bar{a}_x$ be $\bar{G}$, then

$$\bar{G} = \bar{T}^* \bar{T} = (TG^{-1})^* (TG^{-1}).$$
$$= G^{-1} T^* T G^{-1} = G^{-1}$$

$G^{-1}a_j = \bar{a}_j$

So, $$\overline{(\bar{a}_j)} = (\bar{G})^{-1}\bar{a}_j = (\bar{G})^{-1}(G^{-1}a_j) = (\bar{G})^{-1}(\bar{G})a_j$$
$$= a_j.$$

Thus by (i), $$x = \sum \langle x, \bar{a}_j\rangle \overline{(\bar{a}_j)} = \sum \langle x, \bar{a}_j\rangle a_j.$$