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Thm: $P := TS$ is the orthogonal projection of Y ~~into~~ onto the subspace $U := \operatorname{Im}(T)$.

Proof:- Let P_U be an orthogonal projection of Y onto U. We will show that $P \equiv P_U$. We know that $Y = U \oplus U^\perp$, where $U^\perp$ is the orthogonal space of U. Then for $u \in U$ and $v \in U^\perp$ $\exists$ $y \in Y$ such that

$$y = u + v, \text{ where } u = P_U y. \quad \left(\begin{array}{c}\text{as } P_U \text{ is the} \\ \text{orthogonal proj} \\ \text{of } Y \text{ onto } U\end{array}\right).$$

Since $u \in U \Rightarrow \exists$'s $x \in X$ such that

$$u = Tx \in U.$$

$$\Rightarrow \quad Pu = PTx = T\underbrace{ST}_{I_X}x = TI_X x = Tx = u.$$

and, $\because v \in U^\perp$ so

$$\langle x, T^* v\rangle = \langle Tx, v\rangle = 0 \;\; \forall x \in X \; (\because Tx \in U, v \in U^\perp).$$

$$\boxed{S = G^{-1}T^*}$$

$$\Rightarrow \; T^* v = 0 \quad \forall \, v \in U^\perp. \text{ Thus}$$

$$Pv = (TS)v = TG^{-1}T^* v = 0.$$

$$\therefore \; Py = P(u+v) = Pu + Pv = u = P_U y \;\; \forall y \in Y.$$

Thus P is an orthogonal projection of Y onto U.

$\S$ Using G^{-1} define the vectors,

$$\bar{a}_j = G^{-1} a_j \in X \quad j = 1, 2, \ldots, r.$$

The collection $\bar{a}_* = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_r)$ is called the dual frame for the frame a_*. If a_* is a tight frame with frame constant A, then $\bar{a}_*$ is also a tight frame with frame constant $\frac{1}{A}$.