$$= \begin{cases} \geq A\|x\|^2 \\ \leq B\|x\|^2. \end{cases}$$

which proves the theorem.

Dr
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§ Numbers $B \geq A > 0$ are called frame constants of the frame $a_x$. If $B = A$, $a_x$ is a tight frame. i.e., $\|Tx\|^2 = A\|x\|^2$. $\forall x \in X$

Ep. Let $X = \mathbb{C}^2$, then for $r \geq 2$, $\omega = e^{2\pi i/r}$ define

$$a_j = \frac{1}{\sqrt{2}}(\omega^j, \bar{\omega}^j), \quad 0 \leq j \leq r-1 \text{ forms a tight frame}$$

Sol:- Since $X = \mathbb{C}^2$ so, any $x \in X$ is of form $(x_1, x_2) \in X$ so,

$\|x\|^2 = |x_1|^2 + |x_2|^2$, thus we have to show that

$\|Tx\|^2 = A(|x_1|^2 + |x_2|^2)$ or $\langle Tx, Tx\rangle = A(|x_1|^2 + |x_2|^2)$

We know that,

$$Tx = \sum_{j=0}^{r-1} \langle x, a_j\rangle e_j \quad \text{thus}$$

$$\langle Tx, Tx\rangle = \left\langle \sum_j \langle x, a_j\rangle e_j, \sum_k \langle x, a_k\rangle e_k \right\rangle$$

$$= \sum_j \langle x, a_j\rangle \sum_k \overline{\langle x, a_k\rangle} \langle e_j, e_k\rangle$$

$$= \sum_{j=0}^{r-1} \langle x, a_j\rangle \overline{\langle x, a_j\rangle} \qquad (\langle e_j, e_k\rangle = \delta_{jk})$$

Now, $\langle x, a_j\rangle = \frac{1}{\sqrt{2}}(x_1 \bar{\omega}^j + x_2 \omega^j)$. $\qquad \left(\langle x, y\rangle = \sum_i x_i \bar{y}_i\right)$ $\left(\overline{(\bar{\omega}^j)} = \omega^j\right)$

similar

Thus, $\overline{\langle x, q_j \rangle} = \frac{1}{\sqrt{2}} \left( \bar{x}_1 \omega^j + \bar{x}_2 \bar{\omega}^j \right)$.

$$\Rightarrow \quad \|Tx\|^2 = \sum_{j=0}^{r-1} \frac{1}{2} \left( x_1 \bar{\omega}^j + x_2 \omega^j \right) \left( \bar{x}_1 \omega^j + \bar{x}_2 \bar{\omega}^j \right).$$

$$= \frac{1}{2} \left[ \sum_{j=0}^{r-1} x_1 \bar{x}_1 \bar{\omega}^j \omega^j + \sum_{j=0} x_1 \bar{x}_2 (\bar{\omega}^j)^2 + \sum_{j=0}^{r-1} x_2 \bar{x}_1 (\omega^j)^2 + \sum_{j=0}^{r-1} x_2 \bar{x}_2 \omega^j \bar{\omega}^j \right].$$

Now, $\omega^j \bar{\omega}^j = \left(e^{2\pi i/r}\right)^j \left(\overline{e^{2\pi i/r}}\right)^r$

$= 1$

and $\sum_{j=0}^{r-1} (\omega^j)^2 = \frac{1(1-\omega^{2r})}{1-\omega^2} = 0$ $\quad (\because \omega^r = e^{2\pi i} = -1)$.

$\therefore \quad \|Tx\|^2 = \frac{1}{2} \sum_{j=0}^{r-1} \left[ |x_1|^2 + |x_2|^2 \right] = \frac{r}{2} \|x\|^2.$

Hence $q$ is a tight frame with frame constant $r/2$.

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If, $\|Tx\|^2 = A\|x\|^2$, then $\langle Tx, Tx \rangle = A \langle x, x \rangle$

or $\langle Gx, x \rangle = \langle Ax, x \rangle \Rightarrow G = A I_X$, $I_X$ is the identity map.

"How can $x \in X$, be obtained from $y = Tx$"?

Let $a_* = (a_1, a_2, \ldots, a_r)$ be a frame and $G : X \to X$ be a Gram operator. $G^{-1} : X \to X$. Also, $T^* : Y \to X$. So,

$$S := G^{-1} T^* : Y \to X. \text{ Then}$$

$$ST = G^{-1} T^* T = G^{-1} G = I_X$$

$\Rightarrow$ $S$ is the left inverse of $T$.

If $a_*$ is a tight frame then,

$$G^{-1} = \frac{1}{A} I_X$$

So,

$$S = G^{-1} T^* = \frac{1}{A} T^*$$

Thus for a tight frame inverse of $T$ can be obtained without finding the inverse of the matrix.