<u>GRAM OPERATOR</u>: The operator $G : X \to X$, $G = T^*T$, where $T \to$ frame operator and T^* is the adjoint of T, is called the Gram operator. For all $x \in X$

$$G(x) = T^*Tx = T^*\left(\sum_{j=1}^{r} \langle x, a_j \rangle e_j\right)$$
$$= \sum_{j=1}^{r} \langle x, a_j \rangle T^* e_j = \sum_{j=1}^{r} \langle x, a_j \rangle a_j .$$

$\{e_i\}_{i=1}^{r}$ is the basis of $Y := \phi^r$. $T^* e_j = a_j$, $j = 1, 2, \dots r$

<u>Lemma</u>: $\operatorname{Ker} T = \operatorname{Ker} G$.

<u>Proof</u>:- Let $x \in \operatorname{Ker} T$, then $Tx = 0 \Rightarrow Gx = T^*Tx = T^*(0) = 0$.

$\Rightarrow x \in \operatorname{Ker} G \Rightarrow \operatorname{Ker} T \subseteq \operatorname{Ker} G$.

Also, if $x \in \operatorname{Ker} G$ then $T^*Tx = 0$. Then

$$\langle T^*Tx, x \rangle = 0 \Rightarrow \langle Tx, Tx \rangle = 0 \text{ or } \|Tx\|^2 = 0$$

$\therefore Tx = 0 \Rightarrow x \in \operatorname{Ker} T$

$\therefore \operatorname{Ker} G \subseteq \operatorname{Ker} T$

Thus $\operatorname{Ker} G = \operatorname{Ker} T$.

<u>Theorem</u>: A collection $a_0 = (a_1, a_2, \cdots a_r)$ of vectors is a frame for the HS X iff there are constants $B \geq A > 0$ such that

$$A\|x\|^2 \leq \|Tx\|^2 \leq B\|x\|^2. \ \forall x \in X.$$

<u>Proof</u>:- Let $G = T^*T$ be the Gram operator. Then for $x, u \in X$, $\langle x, Gu \rangle = \langle x, T^*Tu \rangle = \langle Tx, Tu \rangle$

$$= \langle T^*Tx, u \rangle = \langle Gx, u \rangle = \langle x, G^*u \rangle$$

$\Rightarrow G = G^*$, hence G is a self-adjoint operator.

$\Rightarrow$ All <u>eigenvalues</u> of G, say λ_i, are real. Hence if λ is an eigenvalue of G and x is the corresponding eigenvector then,

$$\lambda\langle x, x \rangle = \langle Gx, x \rangle = \|Tx\|^2 \geq 0$$

$$\Rightarrow \lambda \geq 0.$$

Thus all eigen values of of G are ~~positive~~ (non-negative)

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and hence can be arranged in increasing order i.e,

$$0 \leq A = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \cdots \leq \lambda_n = B.$$

There is an orthonormal basis $(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$ in X which diagonalizes G, Thus the image of the vector $x = (x_1, x_2, \ldots, x_n) \in X$ is given by,

$$Gx = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3 \ldots, \lambda_n x_n).$$

Hence,

$$\|Tx\|^2 = \langle Gx, x \rangle = \sum (\lambda_i x_i) \bar{x}_i = \sum \lambda_i |x_i|^2$$

$$= \begin{cases} \geq A\|x\|^2 \\ \leq B\|x\|^2. \end{cases}$$

Which proves the theorem.