1 Infinite Products

Let $H(U)$ be the space of all holomorphic functions on an open set U and let (p_j) be a sequence in $H(U)$. Then for each n, $f_n = \prod_{j=1}^{n} p_j$ is holomorphic on U. If the sequence (f_n) converges in $H(U)$ to the function f (say), then $\prod_{j=1}^{\infty} p_j$ is said to be convergent or exists and $f := \prod_{j=1}^{\infty} p_j$ represents an holomorphic function on U. So we shall obtain some sufficient conditions for the infinite product $\prod_{j=1}^{\infty} p_j$ to converge. First we prove a technical result in the form of following Proposition:

Proposition 1 *Given a finite set* $\{u_1, u_2, ..., u_N\}$ *of complex numbers, let* $p_N = \prod\limits_{j=1}^{N} (1 + u_j)$ *and* $p_N^* = \prod\limits_{j=1}^{N} (1 + |u_j|)$. *Then*

(i) $p_N^* \leq \exp(\sum\limits_{j=1}^{N} |u_j|)$,

(ii) $|p_N - 1| \leq p_N^* - 1$.

Proof. (i) Since, $1+|u_j| \leq \exp(|u_j|)$ for each j, we easily prove that $\prod\limits_{j=1}^{N} (1 + |u_j|) = p_N^* \leq \exp\left(\sum\limits_{j=1}^{N} |u_j|\right)$.

(ii) Observe that the result is true for $N = 1$. Let it be true for $k \leq N - 1$ that is $|p_k - 1| \leq p_k^* - 1$. Then

$$\begin{aligned} |p_{k+1} - 1| &= |p_k (1 + u_{k+1}) - 1| = |(p_k - 1)(1 + u_{k+1}) + u_{k+1}| \\ &\leq (p_k^* - 1)(1 + |u_{k+1}|) + |u_{k+1}| = p_{k+1}^* - 1 \end{aligned}$$

which shows that the result is true for $k + 1$ also. Hence the result is true. ■

Remark 1 *The above result will also holds for any finite products* $\prod\limits_{j=M}^{N} (1 + u_j)$ *and* $\prod\limits_{j=M}^{N} (1 + |u_j|)$ *for* $M \leq N$.

Proposition 2 *Let* (u_j) *be a sequence of bounded functions. If* $\sum |u_j|$ *converges uniformly, then* $\prod (1 + u_j)$ *also converges uniformly.*

Proof. By hypothesis $\sum |u_j|$ is uniformly bounded and so is $\exp(\sum |u_j|)$ that is $\exp(\sum |u_j|) < C$ (>0) for all z. Let for each n, $f_n(z) = \prod\limits_{j=1}^{n} (1 + u_j(z))$. Then f_n is holomorphic and for each n and for any z, $|f_n(z)| \leq \prod\limits_{j=1}^{n} (1 + |u_j(z)|) \leq \exp(\sum |u_j|) < C$. Since, the space of all entire functions is complete, we only need to show that the sequence (f_n) is uniformly Cauchy sequence. For $0 < \epsilon <$

1, let n_0 be such that for any $N \geq M \geq n_0$, $\sum_{j=M+1}^{N} |u_j(z)| < \epsilon$ for all z. Then with the use of Proposition 1

$$\begin{aligned}
|f_N - f_M| &= |f_M| \left| \prod_{j=M+1}^{N} (1+u_j) - 1 \right| \leq |f_M| \left(\prod_{j=M+1}^{N} (1+|u_j|) - 1 \right) \\
&\leq |f_M| \left(\exp(\sum_{j=M+1}^{N} |u_j|) - 1 \right) < C\,(\exp(\epsilon) - 1) =: B\ ,
\end{aligned}$$

where $B > 0$. This proves the result. ∎

Proposition 3 *If for each j, $0 \leq u_j < 1$, then $\prod (1-u_j) > 0$ if and only if $\sum u_j < \infty$.*

Proof. Let $f_n = \prod_{j=1}^{n} (1-u_j)$. Then $f_1 \geq f_2 \geq ... \geq 0$ that is (f_n) is a decreasing sequence which is bounded below, so $\lim f_n = f$ exists. If $\sum u_j < \infty$, then by Proposition 2, $\prod_{j=1}^{\infty} (1-u_j) = f > 0$, since each $1 - u_j > 0$. Conversely,

$$0 < f = \prod_{j=1}^{\infty} (1-u_j) \leq ... \leq \prod_{j=1}^{n} (1-u_j) \leq \exp\left(-\sum_{j=1}^{n} u_j \right)$$

and if $\sum u_j = \infty$, then $f = 0$ which gives a contradiction. Hence, $\sum u_j < \infty$. ∎

Proposition 4 *If f_j is entire and not identically zero for each j, and if $\sum |1 - f_j|$ converges uniformly on compact sets, then $f = \prod f_j$ is an entire function.*

Proof. Let $u_j = 1 - f_j$; so $f_j = 1 - u_j$. Then by Propositions 2 and 3, we get the result. ∎

1.1 Weierstrass's Elementary Functions

Functions E_p for any $p = 0, 1, 2, ...$ and for any z, defined by

$$\begin{aligned}
E_0(z) &= 1 - z, E_1(z) = (1-z)\exp(z), ..., \\
E_p(z) &= (1-z)\exp(z + (z^2/2) + ... + (z^p/p)) \qquad (1)
\end{aligned}$$

are called Weierstrass's Elementary Functions. Clearly, these functions are entire functions having precisely one zero at $z = 1$ of multiplicity one. Hence, for any $a \neq 0$, the function $E_p(z/a)$ has a zero at $z = a$ of multiplicity one. We have following Proposition based on the functions $E_p(z)$:

Proposition 5 *Let for any $p = 0, 1, 2, ...$ and for any z, the functions $E_p(z)$ be defined by (1). Then*

(i) $E_p'(z) = -z^p \exp\left(z + (z^2/2) + ... + (z^p/p)\right)$.

(ii) If $E_p(z) = a_0 + a_1 z + ... + a_k z^k + ...$ is a Taylor's expansion of E_p at 0, then $a_0 = 1, a_1 = a_2 = ... = a_p = 0$ and $a_k < 0$ for $k > p$.

(iii) For $|z| \leq 1, |E_p(z) - 1| \leq |z|^{p+1}$.

Proof. On differentiating the expression $E_p(z)$ we directly get the result (i). On equating the series expansion of $E_p(z)$ from (1) and $E_p(z) = a_0 + a_1 z + ... + a_k z^k + ...$, we directly get $a_0 = 1$. From the result (i), we see that E_p' has a zero of multiplicity p at 0. On the other hand by term by term differentiation, we have $E_p'(z) = a_1 + ... + k a_k z^{k-1} + ... + (p+1) a_{p+1} z^p + ...$ Thus on comparing these two expressions, we get the result (ii). Further, from (ii), we have for $|z| \leq 1$,

$$|E_p(z) - 1| \leq \left| \sum_{k=p+1}^{\infty} a_k z^k \right| \leq \sum_{k=p+1}^{\infty} |a_k| \, |z|^k \leq |z|^{p+1} \sum_{k=p+1}^{\infty} (-a_k)$$

since, for $k > p, |a_k| = -a_k$ by (ii). Again, since from (ii) $E_p(1) = 0 = 1 + \sum\limits_{k=p+1}^{\infty} a_k$, we get $\sum\limits_{k=p+1}^{\infty} a_k = -1$ and hence, we get the result (iii). ■

Corollary 1 *For any non-zero* $z_j, |E_p(z/z_j) - 1| \leq |z/z_j|^{p+1}$ *for* $|z| \leq |z_j|$.

Proposition 6 *Let* (z_j) *be a sequence of complex numbers without a limit point and such that* $z_j \neq 0$ *for each* j. *Let* (p_j) *be a sequence of non-negative integers such that* $\sum (r/|z_j|)^{p_j+1}$ *converges for every* $r > 0$. *Then* $P(z) = \prod\limits_j E_{p_j}(z/z_j)$ *is an entire function, with precisely* $z_j's$ *as its zeros, each with the same multiplicity as the number of times it appears in the sequence* (z_j).

Proof. In view of the Corollary 1, for any $z \in \text{cl}B(0,r)$,

$$|E_p(z/z_j) - 1| \leq |z/z_j|^{p+1} \leq (r/|z_j|)^{p_j+1}.$$

Hence, by hypothesis

$$\sum |E_p(z/z_j) - 1| \leq \sum (r/|z_j|)^{p_j+1} < \infty$$

which by Proposition 4 proves the result. ■

The entire function $P(z)$ obtained above has zeros at non-zero z_j. We may construct an entire function $P_1(z) = z^m P(z)$ with zeros at 0 with multiplicity m and at non-zero z_j with prescribed multiplicities. Further, the entire function $P(z)$ obtained above is not the only one having zeros precisely at $z_j's$. If g is any entire function having no zeros, then $f(z) = P(z)g(z)$ is also an entire function having zeros precisely at $z_j's$.

If h is an entire function without any zero, then the function h'/h is also entire and if we define $g(z) = \int_0^z h'/h$, then g is well defined, entire and $h(z) = \exp(g(z))$. We use this fact in the following theorem:

Proposition 7 (Weierstrass's Factorisation Theorem) *Let f be an entire function with $\mathbb{Z}_f = \{z_1, z_2, ..., z_j, ...\}$ as its zero set, each z_j being counted as often as its multiplicity. Let m be the multiplicity of 0 (m may be zero). Then there exist integers $p_1, p_2, ..., p_j, ...$ and an entire function g such that*

$$f(z) = \exp(g(z)) z^m \prod_j^{\infty} E_{p_j}(z/z_j).$$

Proof. We may write $f(z) = z^m f_1(z)$, where f_1 is an entire function. Then zeros of f_1 are precisely the non-zero zeros of f say at $z_1, z_2, ..., z_j, ...$, counted according to their multiplicities. Hence, the function $P(z) = \prod_j E_{p_j}(z/z_j)$ is an entire and has precisely the same zeros as f_1. So the function f_1/P is entire without any zero, hence there is an entire function g such that $f_1/P = \exp(g)$ or $f_1 = P \exp(g)$ which proves the theorem. ■

We apply Weierstrass's Factorisation Theorem in the following example.

Example 1 *Factorise sine function.*

Solution 1 *Consider the function $f(z) = \sin \pi z$. Then f is an entire function with zeros precisely at $n = 0, \pm 1, \pm 2, ...$, each of multiplicity 1. In Weierstrass' s Factorisation Theorem, we have $z_j = n$ and $p_j = 1$. Since, for any $r > 0$, $\sum_{n \neq 0} (r/n)^2$ is convergent and hence, $P_1(z) = \prod_{n \neq 0} (1 - z/n) \exp(z/n)$ is entire having simple zeros precisely at nonzero integers, and*

$$\begin{aligned} P(z) &= z \prod_{n \neq 0} (1 - z/n) \exp(z/n) = z \prod_{n=1}^{\infty} (1 - z/n)(1 + z/n) \exp((z/n) - (z/n)) \\ &= z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right) \end{aligned}$$

is an entire function having zeros of multiplicity one precisely at all integers. Thus according to the Weierstrass' s Factorisation Theorem there exists an entire function g such that

$$\sin \pi z = \exp(g(z)) \ z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right). \tag{2}$$

Now it only remains to determine the function g. On differentiating (2), we get

$$\begin{aligned} \pi \cos \pi z &= \sin \pi z \ g'(z) + \frac{\sin \pi z}{z} + \exp(g(z)) \ z \sum_n \left(-2z/n^2\right) \prod_{k \neq n}^{\infty} \left(1 - z^2/k^2\right) \\ &= \sin \pi z \left[g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} (-2z) / \left(n^2 - z^2\right) \right] \end{aligned}$$

and hence, for all z such that $\sin \pi z \neq 0$, we get

$$\begin{aligned}\pi \cot \pi z &= g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} (2z) \,/\, \left(z^2 - n^2\right) \qquad (3)\\ &= g'(z) + \sum_{n=-\infty}^{\infty} \frac{1}{z-n}\end{aligned}$$

which again on differentiating gives

$$g''(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} - \frac{\pi^2}{\sin^2 \pi z}.$$

The above expression for $g''(z)$ shows that it is periodic with period 1 and for $z = x + iy$ with $0 \leq x \leq 1$ and $|y| > 1$, it is bounded. Hence, by periodicity it is bounded in the entire complex plain and it is an entire function, so by Liouville's Theorem g'' is constant. But $\lim_{y\to\infty} |g''(z)| = 0$. Hence, $g''(z) = 0$ for all z and $g'(z) = c$, a constant. Further, from (3), we observe that $g'(-z) = -g'(z)$, hence, $c = 0$ and g is also a constant, say $\exp(g(z)) = k$. Finally, we get

$$\sin \pi z = k\ z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right)$$

or

$$\frac{\sin \pi z}{\pi\ z} = \frac{k}{\pi} \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right)$$

which on taking $z \to 0$ yields that

$$1 = \frac{k}{\pi}.$$

Thus we get the required factorisation:

$$\sin \pi z = \pi\ z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right).$$

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