Properties of Gamma function

(1) $\Gamma(1) = 1$

Proof: $\Gamma(z) = \dfrac{e^{-\gamma z}}{z\,\phi(z)}$, where $\phi(z) = \prod_{j=1}^{\infty}(1+z/j)\, e^{-z/j}$

Put $z=1$,

$$\Gamma(1) = \frac{e^{-\gamma}}{\phi(1)} \quad \text{and} \quad \phi(1) = e^{-\gamma}$$

(proved earlier)

$\therefore \boxed{\Gamma(1) = 1}$

(2) for any $z \neq 0, -1, -2, \ldots$

$$\Gamma(z+1) = z\,\Gamma(z)$$

In particular, for any positive integer n

$$\Gamma(n+1) = n!$$

Proof Since $\phi(z-1) = \exp(\gamma).\, z\, \phi(z)$

Replace z by $z+1$, we have

$$(z+1)\,\phi(z+1) = e^{-\gamma}\phi(z). \quad \text{——} (1)$$

Since $\Gamma(z) = \dfrac{e^{-\gamma z}}{z\,\phi(z)}$ ——— (2)

$$\therefore \Gamma(z+1) = \frac{e^{-\gamma(z+1)}}{(z+1)\,\phi(z+1)}$$

$$= \frac{e^{-\gamma z}.\, e^{-\gamma}}{e^{-\gamma}\,\phi(z)} \quad \text{from (1)}$$

$$= \frac{e^{-\gamma z}}{\phi(z)} = z\,\Gamma z \quad \text{from (2)}$$

$\therefore \boxed{\Gamma(z+1) = z\,\Gamma z}$

(3) Gauss's formula:

For any $z \neq 0, -1, -2, \dots$

$$\Gamma(z) = \lim_{n} \frac{n!\, n^{z}}{z(z+1)(z+2)\cdots(z+n)}$$

Proof- Since

$$\phi(z) = \prod_{j=1}^{\infty} (1 + z/j) \exp(-z/j)$$

$$= \lim_{n\to\infty} \prod_{j=1}^{n} (1 + z/j) \exp\left(-\frac{z}{j}\right)$$

$$= \lim_{n\to\infty} \exp\left(-\sum_{j=1}^{n} \frac{z}{j}\right) \cdot \prod_{j=1}^{n} \left(1 + \frac{z}{j}\right) \quad \text{—(1)}$$

Also $\Gamma(z) = \dfrac{e^{-\gamma z}}{z\,\phi(z)}$ from (1)

$$= \frac{e^{-\gamma z}}{z \lim_{n\to\infty} \exp\left(-\sum_{j=1}^{n} \frac{z}{j}\right) \cdot \prod_{j=1}^{n} \left(1 + \frac{z}{j}\right)}$$

$$= \frac{e^{-\gamma z}}{z} \cdot \lim_{n\to\infty} \exp\left(\sum_{j=1}^{n} \frac{z}{j}\right) \prod_{j=1}^{n} \left(\frac{j}{z+j}\right)$$

$$= \frac{1}{z} \lim_{n\to\infty} \exp\left(-\gamma z + \sum_{j=1}^{n} \frac{z}{j}\right) \prod_{j=1}^{n} \left(\frac{j}{z+j}\right) \quad \text{—(2)}$$

Now,

$$\lim_{n\to\infty} z\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \gamma\right)$$

$$= \lim_{n\to\infty} z\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n - \gamma + \log n\right)$$

$$= z \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) - \gamma z + \lim_{n\to\infty} z\log n$$

$$= \gamma z - \gamma z + \lim_{n\to\infty} \log n^{z} \qquad \text{(from value of } \gamma\text{)}$$

$$= \lim_{n\to\infty} \log n^{z}. \qquad \text{—— (3)}$$

from (2) and (3), we have

$$\Gamma(z) = \frac{1}{z}\lim_{n\to\infty} \exp(\log n^{z}) \prod_{j=1}^{n} \frac{j}{(z+j)}$$

$$= \lim_{n\to\infty} \frac{n^{z}}{z} \cdot \frac{1.\,2.\,3 \cdots n}{(z+1)(z+2)\cdots(z+n)}$$

$$\boxed{\Gamma(z) = \lim_{n\to\infty} \frac{n^{z}.\, n!}{z(z+1)(z+2)\cdots(z+n)}}$$

(ii) for any z not an integer

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

Proof: Using Gauss's formula (proved above)

$$\Gamma(1-z) = \lim_{n\to\infty} \frac{n^{1-z}.\, n!}{(1-z)(2-z)\cdots(n+1-z)}$$

$$\therefore\ \Gamma(z)\,\Gamma(1-z) = \lim_{n\to\infty} \frac{n^{z}.\, n!}{z(z+1)(z+2)\cdots(z+n)} \cdot \frac{n^{1-z}.\, n!}{(1-z)(2-z)\cdots(n+1-z)}$$

$$= \lim_{n\to\infty} \frac{n}{z}\, \frac{1.2.3.4 \cdots n}{(z+1)(z+2)\cdots(z+n)} \cdot \frac{1.2.3 \cdots n}{(1-z)(2-z)\cdots(n+1-z)}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{z}{n}(1+z)(1+\frac{z}{2})\cdots(1+\frac{z}{n})} \cdot \frac{1}{(1-z)(1-\frac{z}{2})\cdots(1-\frac{z}{n}) \times (n+1-z)}$$

$$\cancel{= \lim_{n\to\infty} \frac{z}{n} \cdot \frac{1}{(n+1-z)} \left\{ \frac{1}{\prod \left(1-\frac{z^{2}}{j^{2}}\right)} \right\}}$$

$$\therefore \frac{1}{\Gamma(z)\,\Gamma(1-z)} = \lim_{n\to\infty} \frac{z}{n}(n+1-z)(1-z^2)\left(1-\frac{z^2}{2^2}\right)\cdots\left(1-\frac{z^2}{n^2}\right)$$

$$= \lim_{n\to\infty} z\left(1+\frac{1-z}{n}\right)\prod_{j=1}^{n}\left(1-\frac{z^2}{j^2}\right)$$

$$= \lim_{n\to\infty} z\prod_{j=1}^{n}\left(1-\frac{z^2}{j^2}\right) = z\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right)$$

Since $\lim_{n\to\infty}\left(1+\frac{1-z}{n}\right) = 1$

and $\sin \pi z = \pi z\prod_{j=1}^{\infty}(1-z^2/j^2)$

$$\therefore \frac{1}{\Gamma(z)\,\Gamma(1-z)} = \frac{\sin \pi z}{\pi}$$

or $\boxed{\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}}$

(5) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Take $z = \frac{1}{2}$, in property (4) (proved above)

$$\cancel{\Gamma\tfrac{1}{2}\,\Gamma(1-\tfrac{1}{2})}\ \left(\Gamma\tfrac{1}{2}\right)^2 = \frac{\pi}{\sin \pi/2}$$

or $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

<u>Proposition</u> for a complex number z such that $Re(z) > 1$,

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt.$$

<u>Proof</u>: Let f_n be the real valued function on $(0,\infty)$ defined by

$$\cancel{f_n(t) = (1-\tfrac{t}{n}}$$

$$f_n(t) = \begin{cases} (1-\frac{t}{n})^n & \text{for } 0<t\le n, \\ 0, & \text{for } t>n \end{cases} \quad (n=1,2,\dots)$$

Each f_n is measurable and

$$|f_n(t)| \le \exp(-t), \text{ for all } t \ge 0 \text{ and for all } n.$$

Hence for any $x>1$,

$$|f_n(t).\, t^{x-1}| \le \exp(-t).\, t^{x-1}, \forall t \ge 0 \text{ and for all } n$$

$\therefore$ By dominated convergence theorem

$$\lim_n \int_0^\infty f_n(t).\, t^{x-1}\, dt = \int_0^\infty \lim_n f_n(t).\, t^{x-1}\, dt$$

$$= \int_0^\infty e^{-t}.\, t^{x-1}\, dt.$$

But for each n,

$$\int_0^\infty f_n(t).\, t^{x-1}\, dt = \int_0^n (1-\tfrac{t}{n})^n.\, t^{x-1}\, dt.$$

integrating by parts, we get

$$\int_0^n \left(1-\frac{t}{n}\right)^n \cdot t^{x-1}\,dt = \frac{n!\; n^x}{x(x+1)(x+2)\cdots(x+n)}$$

$$\therefore \lim_n \int_0^\infty f_n(t)\cdot t^{x-1}\,dt = \lim_{n\to\infty} \frac{n!\; n^x}{x(x+1)(x+2)\cdots(x+n)}$$

$$= \Gamma(n), \text{ (by Gauss's formula) for } x>1$$

$\therefore$ the gamma function coincides with the holomorphic function $\varphi(z) = \int_0^\infty e^{-t}\cdot t^{z-1}\,dt$.

on the set $L = \{z : z \text{ real}, z>1\}$.

Since the gamma function is also holomorphic on $U = \{z : Re(z)>1\}$ and subset L of U contains a limit point of U.

$\Rightarrow$ the gamma function coincides with φ on U.

Legendre's duplication formula

~~Since~~ →

$$2^{2z-1}\Gamma z\,\Gamma(z+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2z)$$

Proof Since $\Gamma(z) = \dfrac{e^{-\gamma z}}{z\,\phi(z)}$

Taking log on both sides, we get

$$\log\Gamma(z) = -\gamma z - \log z - \log\phi(z)$$

where $\phi(z) = \prod_{\delta=1}^{\infty}(1+z/\delta)\,e^{-z/\delta}$

$$\therefore \log\Gamma(z) = -\gamma z - \log z - \sum_{\delta=1}^{\infty}\left\{\log(1+\tfrac{z}{\delta}) - \tfrac{z}{\delta}\right\}$$

Differentiate w.r.t. z, we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{\delta=1}^{\infty}\left\{\frac{1/\delta}{1+z/\delta} - \frac{1}{\delta}\right\}$$

$$= -\gamma - \frac{1}{z} - \sum_{\delta=1}^{\infty}\left\{\frac{1}{z+\delta} - \frac{1}{\delta}\right\}$$

Again, differentiate w.r.t. z, we get

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \frac{1}{z^2} + \sum_{\delta=1}^{\infty}\frac{1}{(z+\delta)^2} \quad (\because \gamma \text{ is constant})$$

$$= \sum_{\delta=0}^{\infty}\frac{1}{(z+\delta)^2} \qquad \text{——— (1)}$$

Replace z by $z+\frac{1}{2}$, we have

$$\frac{d}{dz}\left(\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+\frac{1}{2}+j)^2} \quad \text{--- (2)}$$

Adding (1) and (2), we get

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) + \frac{d}{dz}\left(\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\right)$$

$$= \sum_{j=0}^{\infty} \frac{1}{(z+j)^2} + \sum_{j=0}^{\infty} \frac{1}{(z+j+\frac{1}{2})^2}$$

$$= 4 \sum_{j=0}^{\infty} \frac{1}{4(z+j)^2} + \sum_{j=0}^{\infty} \frac{4}{(2z+2j+1)^2}$$

$$= 4 \left[\sum_{j=0}^{\infty} \left\{ \frac{1}{(2z+2j)^2} + \frac{1}{(2z+2j+1)^2} \right\} \right]$$

$$= 4 \sum_{n=0}^{\infty} \frac{1}{(2z+n)^2} \quad \text{--- (3)}$$

Similarly $\Gamma(2z) = \dfrac{e^{-2\gamma z}}{2z\,\phi(2z)}$ ——— (4)

where $\phi(2z) = \prod_{j=1}^{\infty} \left(1+\frac{2z}{j}\right) e^{-2z/j}$ —(5)

Taking log on both sides in (4) (using (5))

$$\log \Gamma(2z) = -2\gamma z - \log(2z) - \log \phi(2z),$$

$$= -2\gamma z - \log(2z) - \sum_{j=1}^{\infty} \left\{ \log\left(1+\frac{2z}{j}\right) - \frac{2z}{j} \right\}$$

Differentiate it w.r.t z, we get

$$2\frac{\Gamma'(2z)}{\Gamma(2z)} = -2\gamma - \frac{2}{2z} - \sum_{j=1}^{\infty}\left\{\frac{2/j}{1+2z/j} - \frac{2}{j}\right\}$$

Again, differentiate w.r.t, z, we have

$$2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right) = \frac{1}{z^2} + \sum_{j=1}^{\infty}\frac{4}{(2z+j)^2}$$

$$= \sum_{j=0}^{\infty}\frac{4}{(2z+j)^2}$$

or $$2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right) = \sum_{n=0}^{\infty}\frac{4}{(2z+n)^2} \quad \text{——(6)}$$

from (3) and (6), we have

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) + \frac{d}{dz}\left(\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\right) = 2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right)$$

Integrating it w.r.t. 'z', we have

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2\frac{\Gamma'(2z)}{\Gamma(2z)} + C_1 \quad \text{——(7)}$$

where C_1 is constant of integration.

Again integrate (7), w.r.t 'z',

$$\log\Gamma(z) + \log\left(\Gamma(z+\tfrac{1}{2})\right) = 2\log\Gamma(2z) + C_1 z + C_2$$

where C_2 is constant.

$$\therefore \Gamma(z)\,\Gamma(z+\tfrac{1}{2}) = \left(\Gamma(2z)\right)^{2}\cdot e^{C_1 z + C_2} \quad \text{——(8)}$$

Put $z = \frac{1}{2}$ in (8), we get

$$\Gamma\tfrac{1}{2}\,\Gamma 1 = (\Gamma 1)\, e^{\frac{c_1}{2} + c_2}$$

or $$\sqrt{\pi} = e^{\frac{c_1}{2} + c_2} \quad \text{——(9)}$$

Now put $z = 1$ in (8), we get

$$\Gamma 1\, \Gamma\tfrac{3}{2} = (\Gamma 2)\, e^{c_1 + c_2}$$

or $$\frac{\sqrt{\pi}}{2} = e^{c_1 + c_2} \quad \text{——(10)}$$

$$\left(\because \Gamma\tfrac{3}{2} = \tfrac{1}{2}\Gamma\tfrac{1}{2} = \tfrac{\sqrt{\pi}}{2}\right)$$

Dividing (9) by (10), we get

$$\frac{\sqrt{\pi}}{\sqrt{\pi}/2} = \frac{e^{\frac{c_1}{2} + c_2}}{e^{c_1 + c_2}}$$

or $$e^{\frac{c_1}{2}} = \frac{1}{2}$$

or $$c_1 = -2\log 2$$

Putting c_1 in (9), we get

$$e^{c_2} = \sqrt{\pi}\, e^{\log 2}$$

$$= 2\sqrt{\pi}$$

or $$c_2 = \log(2\sqrt{\pi})$$

Putting c_1 & c_2 in, (8), we get

$$\Gamma(z)\,\Gamma(z + \tfrac{1}{2}) = \Gamma(2z)\, e^{-2z\log 2 + \log(2\sqrt{\pi})}$$

$$\Gamma z\,\Gamma(z+\tfrac{1}{2}) = \Gamma(2z)\, e^{\log(2^{-2z}\cdot 2\sqrt{\pi})}$$

or $$\Gamma z\,\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$$

or $$\boxed{2^{2z-1}\Gamma z\,\Gamma(z+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2z)}$$

Proved

Relation between Gamma and Zeta function

Prove that

$$\Gamma(z)\,\zeta(z) = \int_0^\infty \frac{t^{z-1}}{e^t-1}\,dt$$

Proof Since $Re(z)>0$,

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt.$$

Changing 't' to 'nt' for a positive integer 'n', we get

$$\Gamma(z) = \int_0^\infty e^{-nt}\,(nt)^{z-1}\,d(nt)$$

$$= n^z \int_0^\infty e^{-nt}\, t^{z-1}\, dt$$

$$\frac{\Gamma(z)}{n^z} = \int_0^\infty e^{-nt}\, t^{z-1}\, dt$$

$$\therefore \sum_{n=1}^{\infty} \frac{\Gamma(z)}{n^z} = \sum_{n=1}^{\infty}\int_0^\infty e^{-nt}\, t^{z-1}\, dt \quad \text{for } Re(z)>1$$

$$\Rightarrow \Gamma(z)\sum_{n=1}^{\infty}\frac{1}{n^z} = \int_0^\infty\Big(\sum_{n=1}^{\infty} e^{-nt}\Big)\, t^{z-1}\, dt$$

$$\Gamma(z)\,\zeta(z) = \int_0^\infty (e^{-t} + e^{-2t} + \dots\dots)\, t^{z-1}\, dt$$

$$= \int_0^\infty \frac{e^{-t}}{1-e^{-t}} \cdot t^{z-1}\, dt$$

$$= \int_0^\infty \frac{1}{e^{t}-1} \cdot t^{z-1}\, dt$$

$$\Rightarrow \boxed{\Gamma(z)\,\zeta(z) = \int_0^\infty \frac{t^{z-1}}{e^{t}-1}\, dt}$$

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