Unit - II

Linear Systems- Let us consider a system of first order differential equations of the form

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y) \qquad (1)$$

Where t is an independent variable. And x & y are dependent variables.

The system (1) is called a linear system if both F(x, y) and G(x, y) in x and y.

Also system (1) can be written as

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t)$$
$$\frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t) \qquad (2)$$

Where $a_i(t)$, $b_i(t)$ and $a_i(t)$ $\forall\, i = 1, 2$ are continuous functions on $[a, b]$.

Homogeneous and Non-Homogeneous Linear Systems- The system (2) is called a homogeneous linear system, if both $f_1(t)$ and $f_2(t)$ are identically zero and if both $f_1(t)$ and $f_2(t)$ are not equal to zero, then the system (2) is called a non-homogeneous linear system.

Solution- A pair of functions $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ defined on $[a, b]$ is said to be a solution of (2) if it satisfies (2).

Example-

$$\frac{dx}{dt} = 4x - y \quad ....A$$
$$\frac{dy}{dt} = 2x + y \quad ....B \qquad (3)$$

From A, $y = 4x - \frac{dx}{dt}$ putting in B we obtain $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$ is a 2nd order differential equation. The auxiliary equation is $m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$ so $\begin{matrix} x = e^{2t} \\ x = e^{3t} \end{matrix}$ putting $x = e^{2t}$ in A, we obtain $y = 2e^{2t}$ again putting $x = e^{3t}$ in A, we obtain $y = e^{3t}$. Therefore the solutions of (3) are

$$\begin{matrix} x = e^{2t} \\ y = 2e^{2t} \end{matrix} \text{ and } \begin{matrix} x = e^{3t} \\ y = e^{3t} \end{matrix} \qquad (4)$$

Theorem-1 If t_0 is any point of $[a,b]$ and x_0 & y_0 are any two numbers, then the system (2) has a unique solution $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ with $\begin{matrix} x(t_0) = x_0 \\ y(t_0) = y_0 \end{matrix}$.

Theorem-2 If the homogeneous system

$$\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned} \qquad (5)$$

has two solutions $\begin{matrix} x = x_1(t) \\ y = y_1(t) \end{matrix}$ and $\begin{matrix} x = x_2(t) \\ y = y_2(t) \end{matrix}$ (6)

on $[a,b]$. Then $\begin{matrix} x = c_1x_1(t) + c_2x_2(t) \\ y = c_1y_1(t) + c_2y_2(t) \end{matrix}$ (7)

is also a solution of (5) on $[a,b]$ for any two constants c_1 and c_2.

Theorem-3 If the two solutions $\begin{matrix} x = x_1(t) \\ y = y_1(t) \end{matrix}$ and $\begin{matrix} x = x_2(t) \\ y = y_2(t) \end{matrix}$ (6) of the homogeneous system (5) have a wronskian $W(t)$ that does not vanish on $[a,b]$, then $\begin{matrix} x = c_1x_1(t) + c_2x_2(t) \\ y = c_1y_1(t) + c_2y_2(t) \end{matrix}$ (7) is a general solution of homogeneous system (5) on $[a,b]$.

Note- The wronskian W(t) of the solutions (4) is

$$W(t) = \begin{vmatrix} e^{3t} & e^{2t} \\ e^{3t} & 2e^{2t} \end{vmatrix}$$

$$= e^{5t}$$

Theorem -4 The wronskian W(t) of two solutions (6) of homogeneous system (5) is either identically zero or nowhere zero on $[a,b]$ i.e

$W(t) = 0$ (linearly dependent) or $W(t) \neq 0$ (linearly independent).

The wronskian W(t) satisfies the differential equation, $\frac{dW}{dt} = [a_1(t) + b_2(t)]W$ and on integrating between the limits 0 to t we obtain

$$W(t) = ce^{\int_0^t [a_1(t) + b_2(t)]dt}.$$

Theorem -5 If the two solutions $\begin{matrix} x = x_1(t) \\ y = y_1(t) \end{matrix}$ and $\begin{matrix} x = x_2(t) \\ y = y_2(t) \end{matrix}$ of homogeneous system (5) are linearly independent on $[a, b]$ and if $\begin{matrix} x = x_p(t) \\ y = y_p(t) \end{matrix}$ is any particular solution of non-homogeneous system (2) on $[a, b]$, then $\begin{matrix} x = c_1 x_1(t) + c_2 x_2(t) + x_p(t) \\ y = c_1 y_1(t) + c_2 y_2(t) + y_p(t) \end{matrix}$ is a general solution of of non-homogeneous system (2) on $[a, b]$.

Example- Show that $\begin{matrix} x = e^{4t} \\ y = e^{4t} \end{matrix}$ and $\begin{matrix} x = e^{-2t} \\ y = -e^{-2t} \end{matrix}$ are the solutions of the homogeneous system

$\begin{matrix} \dfrac{dx}{dt} = x + 3y \\ \dfrac{dy}{dt} = 3x + y \end{matrix}$ and find the particular solution $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ of the given system for which $x(0) = 5$ and $y(0) = 1$.

Solution- Let

$$\begin{matrix} \dfrac{dx}{dt} = x + 3y \\ \dfrac{dy}{dt} = 3x + y \end{matrix} \qquad (1)$$

First, we show that each of the pair $\begin{matrix} x = e^{4t} \\ y = e^{4t} \end{matrix}$ and $\begin{matrix} x = e^{-2t} \\ y = -e^{-2t} \end{matrix}$ satisfy the system (1). In order to determines a particular solution of (1), let us consider $\begin{matrix} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{matrix}$ (2) be a particular solution of (1), where the constants c_1 and c_2 are to be determined. Putting the values of $x_1(t) = e^{4t}$, $x_2(t) = e^{-2t}$, $y_1(t) = e^{4t}$ and $y_2(t) = -e^{-2t}$ in (2) and using the given conditions $x(0) = 5$ and $y(0) = 1$, we obtain $c_1 = 3$ and $c_2 = 2$.

Therefore $\begin{matrix} x = 3e^{4t} + 2e^{-2t} \\ y = 3e^{4t} - 2e^{-2t} \end{matrix}$ is a particular solution.

Example Show that $\begin{matrix} x = 3t - 2 \\ y = -2t + 3 \end{matrix}$ is a particular solution of the non-homogeneous system

$$\begin{aligned}\frac{dx}{dt} &= x + 2y + t - 1 \\ \frac{dy}{dt} &= 3x + 2y - 5t - 2\end{aligned}$$ and write the general solution of this system.

Hint- Let $$\begin{aligned}\frac{dx}{dt} &= x + 2y + t - 1 \\ \frac{dy}{dt} &= 3x + 2y - 5t - 2\end{aligned} \qquad (1)$$

Now $\begin{aligned}x &= 3t - 2 \\ y &= -2t + 3\end{aligned}$ will be a particular solution of the non-homogeneous system (1) if it satisfies the system (1). In order to find a general solution of system (1), we have to find a solution corresponding homogeneous system $\begin{aligned}\frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 3x + 2y\end{aligned}$ (2) to system (1) as similar in example in equation (3).

Answer- $$\begin{aligned}x &= 2c_1e^{4t} + c_2e^{-t} + 3t - 2 \\ y &= 3c_1e^{4t} - c_2e^{-t} - 2t + 3\end{aligned}$$

Homogeneous Linear Systems with Constant Coefficients- Let us consider a homogeneous linear system with constant coefficients $$\begin{aligned}\frac{dx}{dt} &= a_1x + b_1y \\ \frac{dy}{dt} &= a_2x + b_2y\end{aligned} \qquad (1)$$

Where a_1, b_1, a_2 and b_2 are constants. Suppose $\begin{aligned}x &= Ae^{mt} \\ y &= Be^{mt}\end{aligned}$ (2)

(where A, B and m are to be determined) be a solution of the system (1), then it satisfies (1) so

$$\begin{aligned}Ame^{mt} &= (a_1A + b_1B)e^{mt} \\ Bme^{mt} &= (a_2A + b_2B)e^{mt}\end{aligned}$$

Or

$$\begin{aligned}(a_1 - m)A + b_1B &= 0 \\ a_2A + (b_2 - m)B &= 0\end{aligned} \qquad (3)$$

is a system of equations of the form $ax = 0$ has a trivial solution $x = 0$, if $A = B = 0$ so for a nontrivial solution $x \neq 0$ of (3), we have $a = 0$ i.e

$\begin{vmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{vmatrix} = 0$, on expanding we obtain a quadratic equation in m

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0 \qquad (4)$$

gives two values of m say m_1 and m_2. Now the following three cases arise

Case-1 If m_1 and m_2 are real and distinct, then corresponding to m_1, we find the values of A and B say A_1 and B_1 by equation (3), so the first nontrivial solution is $\begin{matrix} x = A_1 e^{m_1 t} \\ y = B_1 e^{m_1 t} \end{matrix}$. Similarly corresponding to m_2, we find the another nontrivial solution $\begin{matrix} x = A_2 e^{m_2 t} \\ y = B_2 e^{m_2 t} \end{matrix}$

Therefore the general solution is $\begin{matrix} x = c_1 (A_1 e^{m_1 t}) + c_2 (A_2 e^{m_2 t}) \\ y = c_1 (B_1 e^{m_1 t}) + c_2 (B_2 e^{m_2 t}) \end{matrix}$

Example- Find the general solution of the system of equations

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dy} = 4x - 2y$$

Solution- Let $\begin{matrix} \frac{dx}{dt} = x + y \\ \frac{dy}{dy} = 4x - 2y \end{matrix}$ (1)

On comparing $a_1 = 1, b_1 = 1, a_2 = 4$ and $b_2 = -2$, the auxiliary equation is $m^2 + m - 6 = 0$ gives $m = -3,\ 2$

Where A and B satisfy $\begin{matrix} (1 - m)A + B = 0 \\ 4A + (-2 - m)B = 0 \end{matrix}$ (2)

When $m = -3$, then by (2) we get $A = 1,\ B = -4$ and the first nontrivial solution is $\begin{matrix} x = e^{-3t} \\ y = -4e^{-3t} \end{matrix}$.

Similarly for $m=2$, then by (2) we get $A=1, B=1$ and the another nontrivial solution is

$$\begin{aligned} x &= e^{2t} \\ y &= 4e^{2t} \end{aligned}.$$

Therefore the general solution is $\begin{aligned} x &= c_1 e^{-3t} + c_2 e^{2t} \\ y &= -4c_1 e^{-3t} + c_2 e^{2t} \end{aligned}$

Example- Find the general solution of the system $\begin{aligned} \frac{dx}{dt} &= -3+4y \\ \frac{dy}{dy} &= -2x+3y \end{aligned}$

Answer $\begin{aligned} x &= 2c_1 e^{-t} + c_2 e^{t} \\ y &= c_1 e^{-t} + c_2 e^{t} \end{aligned}$

Case-2 If m_1 and m_2 are conjugate complex numbers of the form $a \pm ib$, where a and b are real numbers with $b \neq 0$, then we consider two linearly independent solutions $\begin{aligned} x &= A_1^* e^{(a+ib)t} \\ y &= B_1^* e^{(a+ib)t} \end{aligned}$ (1) and $\begin{aligned} x &= A_2^* e^{(a-ib)t} \\ y &= B_2^* e^{(a-ib)t} \end{aligned}$, where $A_1^* = A_1 + iA_2$, $B_1^* = B_1 + iB_2$, $A_2^* = A_1 - iA_2$ and $B_2^* = B_1 - iB_2$ resp. Putting the values of A_1^* and B_1^* in (1), we have

$$\begin{aligned} x &= (A_1 + iA_2)e^{at}(\cos bt + i \sin bt) \\ y &= (B_1 + iB_2)e^{at}(\cos bt + i \sin bt) \end{aligned}$$

Or

$$\begin{aligned} x &= e^{at}\left[(A_1 \cos bt - A_2 \sin bt) + i(A_1 \sin bt + A_2 \cos bt)\right] \\ y &= e^{at}\left[(B_1 \cos bt - B_2 \sin bt) + i(B_1 \sin bt + B_2 \cos bt)\right] \end{aligned}$$

Equating real and imaginary parts, we obtain two linearly independent solutions say

$$\begin{aligned} x &= e^{at}(A_1 \cos bt - A_2 \sin bt) \\ y &= e^{at}(B_1 \cos bt - B_2 \sin bt) \end{aligned} \quad (3) \quad \text{and} \quad \begin{aligned} x &= e^{at}(A_1 \sin bt - A_2 \cos bt) \\ y &= e^{at}(B_1 \sin bt - B_2 \cos bt) \end{aligned} \quad (4)$$

Therefore the general solution is

$$\begin{aligned} x &= e^{at}\left[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)\right] \\ y &= e^{at}\left[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)\right] \end{aligned}$$

Example- $\frac{dx}{dt} = 4x - 2y$
$\frac{dy}{dt} = 5x + 2y$

Hint- $\frac{dx}{dt} = 4x - 2y$
$\frac{dy}{dt} = 5x + 2y$ (1)

The auxiliary equation is $m^2 - 6m + 18 = 0$ gives $m = 3 \pm 3i$, taking a nontrivial solution $x = (A_1 + iA_2)e^{3t}(\cos 3t + i\sin 3t)$, $y = (B_1 + iB_2)e^{3t}(\cos 3t + i\sin 3t)$ (2) of (1), where A_1, B_1, A_2 and B_2 are to be determined. For this (2) satisfies (1) and equating the coefficients of $\cos 3t$ and $\sin 3t$ on both sides.

Answer- $x = e^{3t}(2c_1\cos 3t + 2c_2\sin 3t)$
$y = e^{3t}\left[c_1(\cos 3t + 3\sin 3t) + c_2(\sin 3t - 3\cos 3t)\right]$

Case -3 If $m_1 = m_2 = m$ are equal roots then we should have only one linearly solution

$x = Ae^{mt}$, $y = Be^{mt}$ and the 2nd linearly independent solution will be of the form $x = Ate^{mt}$, $y = Bte^{mt}$. But actually, we consider the 2nd linearly independent solution

$x = (A_1 + A_2t)e^{mt}$, $y = (B_1 + B_2t)e^{mt}$, where A, B, A_1, A_2, B_1 and B_2 are to be determined.

Therefore the general solution is $x = c_1Ae^{mt} + c_2(A_1 + A_2t)e^{mt}$
$y = c_1Be^{mt} + c_2(B_1 + B_2t)e^{mt}$

Example- Find the general solution of the system

$$\frac{dx}{dt} = 3x - 4y$$
$$\frac{dy}{dt} = x - y$$

Solution- Let $$\frac{dx}{dt} = 3x - 4y$$ $$\frac{dy}{dt} = x - y \qquad (1)$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1$$

Let $$x = Ae^t$$ $$y = Be^t \qquad (2)$$

be a solution of (1), where A and B satisfy

$$\begin{aligned} 2A - 4B &= 0 \\ A - 2B &= 0 \end{aligned}$$ gives $A = 2,\ B = 1$,so

$$\begin{aligned} x &= 2e^t \\ y &= e^t \end{aligned} \qquad (3)$$

be a first linearly independent solution of (1). We consider the second linearly independent solution of (1) of the form $$\begin{aligned} x &= (A_1 + A_2 t)e^t \\ y &= (B_1 + B_2 t)e^t \end{aligned} \qquad (4)$$

so it satisfies (1)

$$\begin{aligned} (2A_1 - A_2 - 4B_1) + (2A_1 - 4B_2)t &= 0 + 0t \\ (A_1 - 2B_1 - B_2) + (A_2 - 2B_2)t &= 0 + 0t \end{aligned}$$ on equating both sides we have

$$\begin{aligned} 2A_1 - A_2 - 4B_1 &= 0 \\ 2A_1 - 4B_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} A_1 - 2B_1 - B_2 &= 0 \\ A_2 - 2B_2 &= 0 \end{aligned} \qquad (5)$$

On solving the equations in (5), we obtain $A_1 = 1, B_1 = 0,\ A_2 = 2$ & $B_2 = 1$

The another linearly independent solution is

$$\begin{aligned} x &= (1 + 2t)e^t \\ y &= te^t \end{aligned}$$

Therefore the general solution is

$$x = 2c_1e^t + c_2(1+2t)e^t$$
$$y = c_1e^t + c_2te^t$$

Example- Find the general solution of the system

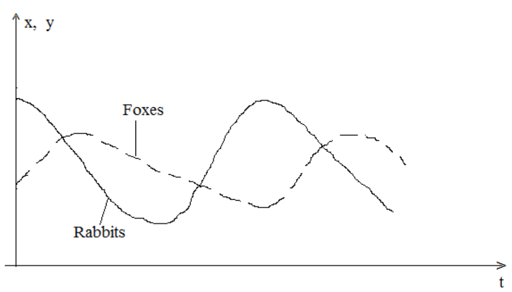
$$\frac{dx}{dt} = 5x + 4y$$
$$\frac{dy}{dt} = -x + y$$

Answer $x = -2c_1e^{3t} + c_2(1+2t)e^{3t}$
$y = c_1e^{3t} - c_2te3t$

Non-Linear Systems: Volterra's Prey- Predator Equations-

Everyone knows that there is a constant struggle for survival among different species of animals living in the same environment. One kind of animal survives by eating another and a second by

For an example of this universal conflict between the predator and its prey, let us imagine an island inhabited by foxes and rabbits. The foxes eat rabbits and the rabbits eat clovers. Let us assume that there is so much clovers then the rabbits have an ample supply of food. When the rabbits are abundant, then the foxes flourish and their population grows. When the foxes become too numerous and eat too many rabbits, then they enter into a period of famine and their population begins to decline. As the foxes decrease, then the rabbits become relatively safe and their population starts to increase again. Thus we have an endless repeated cycle of the increase and decrease in two species of animals and the fluctuations in two species are given by the following figure



If x and y are the number of rabbits and foxes at any time t, then in the presence of an unlimited supply of clovers,

The rate of change of rabbits is $\frac{dx}{dt} = ax,\ a > 0$, after some encounter between the rabbits and foxes the rate of change of rabbits is $\frac{dx}{dt} = ax - bxy,\ a, b > 0$ (1)

In the absence of rabbits the foxes die and the rate of change of foxes is $\frac{dy}{dt} = -cy,\ c > 0$ and after some encounter of foxes with rabbits their population grows and the rate of change of foxes become

$$\frac{dy}{dt} = -cy + dxy, \quad c, d > 0 \tag{2}$$

These two equations are called the volterra's prey-predator equations.

For the solution of these equations, we divide (2) by (1)

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-y(c - dx)}{x(a - by)}$$

Or

$$\frac{dy}{dx} = \frac{-y(c - dx)}{x(a - by)} \tag{3}$$

on separating the variables, we have

$$\frac{(c - dx)dx}{x} + \frac{(a - by)dy}{y} = 0$$

$$\int \left(\frac{c}{x} - d \right) dx + \int \left(\frac{a}{y} - b \right) dy = 0$$

On integrating, we have

$$\begin{aligned} &c \log x + a \log y = dx + by + \log K \\ &or \quad x^c y^a = K e^{(dx + by)} \end{aligned} \tag{4}$$

In order to determine K putting $x(t_0) = x_0,\ y(t_0) = y_0$ in (4) so

$$K = x_0^c y_0^a e^{-(d\,x_0 + b\,y_0)}$$

Therefore the solution of volterra’s prey- predator equations is

$$x^c y^a = \left(x_0^c y_0^a e^{-(d\,x_0 + b\,y_0} \right) e^{(d\,x + b\,y)}$$