# —: Simplex Method :—

Any linear programming problem involving two variables can be easily solved with the help of graphical method as it is easier to deal with two dimensional graph. All the feasible solutions in graphical methods lies within the feasible area on the graph and we used to test the corner points of the feasible ~~solution~~ area for the optimum solution. We used to test all the corner points by putting these values in the objective function.

But if number of variables increase from two, it becomes very difficult to solve the problem by drawing its graph as the problem becomes too complex.

Simplex method is a technique to solve the LPP ~~which~~ that was developed by G.B. Dantzig, an American mathematician. This method is mathematical treatment of the graphical method. Here also various corner points of the feasible area are tested for optimality. But it is not possible to test all the corner points since number of corner points increases manifolds as the numbers of equations and variables increases. Therefore, in simplex method the number of corner points to be tested is reduced considerably by using a very effective algorithm which leads us to optimal solution corner point in only a few itterations.

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Let us take one example and proceed step by step:

Problem:- 1:

Objective function is to maximise $Z = 12x_1 + 15x_2 + 14x_3$ subjected to conditions

$$-x_1 + x_2 \leq 0 \quad \text{——— (1)}$$
$$-x_1 + 2x_3 \leq 0 \quad \text{——— (2)}$$
$$x_1 + x_2 + x_3 \leq 100 \quad \text{——— (3)}$$
$$x_1, x_2, x_3 \geq 0$$

Solution:-

Step 1:— Right hand side of all the constraints must be either zero or +ve. If they are −ve they must be made +ve by multiplying both side by -1 and sign of inequality would be reversed.

All the inequalities are converted into equalities by adding or substracting slack or surplus varibles. These If constraint is $\leq$, the slack variables are added If constraint is $\geq$, then surplus variable is substracted. In this problem slack variable $S_1, S_2, S_3$ are added in three equalities:

$$-x_1 + x_2 + S_1 = 0$$
$$-x_1 + 2x_3 + S_2 = 0$$
$$x_1 + x_2 + x_3 + S_3 = 100$$
$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

And objective function becoms

maximise $Z = 12x_1 + 15x_2 + 14x_3 + 0.S_1 + 0.S_2 + 0.S_3$

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Step-2 :- We start with a feasible solution and then move towards optimal solution in next iteration.

* Initial feasible solution is preferably chosen to be the origin i.e. $x_1, x_2, x_3$ assume to be zero.

$$x_1, x_2, x_3 = 0$$

from this we get $s_1 = 0$, $s_2 = 0$ and $S_3 = 100$. from

Basic variables are the variables which are presently in the solution i.e. $s_1, s_2$ & $s_3$. Non basic variables are the variables which are set to be zero and are not in the current solution e.g. $x_1, x_2$ & $x_3$.

The above information can be expressed in the table

Table 1

| | objective function coefficent $(C_j)$ | 12 | 15 | 14 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $C_i$ | Basic variables (B.V.) | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
| 0 | $s_1$ | −1 | 1 | 0 | 1 | 0 | 0 | 0 |
| 0 | $s_2$ | −1 | 0 | 2 | 0 | 1 | 0 | 0 |
| 0 | $s_3$ | 1 | 1 | 1 | 0 | 0 | 1 | 100 |

In the above table, $I^{st}$ row represents coefficients of objective function, $2^{nd}$ row represent different variables (first regular variables and then slack/surplus). $3^{rd}$, $4^{th}$ and $5^{th}$ row represents coefficients of variables in all the constraints.

$I^{st}$ column represents coefficients of basic variables in the objective function, $2^{nd}$ column represent basic variables and last column represent RHS of the constraints in standard form.

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In table 1 : Current solution is $s_1=0,\ s_2=0,\ s_3=100$

Whenever any basic variable assumes zero variable, the current solution is said to be degenerate, as in present problem $s_1=0$ & $s_2=0$ the proble can be further solved by substituting $s_1=t$ and $s_2=t$ where t is very small +ve number.

Step 3 : Optimality test :-

Optimatility test can be performed to find whether current solution is optimal or not. For this make a new ~~row~~ the last row in the form of Ej.

where $\boxed{E_j = \sum e_i \cdot a_{ij}}$

where $a_{ij}$ – coeffecients in the body identity matrix for $i^{th}$ row & $J^{th}$ column.

Now make a new last row $\boxed{c_j = C_j - E_j}$

Table 2.

| | $C_j$ | 12 | 15 | 14 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $e_i$ | B.V. | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
| 0 | $s_1$ | −1 | 1 | 0 | 1 | 0 | 0 | t |
| 0 | $s_2$ | −1 | 0 | 2 | 0 | 1 | 0 | t |
| 0 | $s_3$ | 1 | 1 | 1 | 0 | 0 | 1 | 100 |
| $E_j = \sum e_i a_{ij}$ | | 0 | 0 | 0 | 0 | 0 | 0 | |
| $c_j = C_j - E_j$ | | 12 | 15 ↑ key column | 14 | 0 | 0 | 0 | |

If any of the value of $(C_j - E_j)$ is +ve then it means that most +ve value will give the key column. In the present case $x_2$ is potential variable (most +ve value).

If all the values in $(C_j - E_j)$ row is −ve then it means optimum solution has been reached.

⇒ **Step 4 :- Iterate towards optimum solution :-**

Maximum value of $C_j - E_j$ gives the key column.

Now find out key row. For this first introduce ratio column at the last. To obtained the ratio column, devide the coefficients in RHS column by the corresponding coefficients in the key column.

Now look for the least +ve value in the ratio column and that would give the key row.

Key element

| $e_i$ | $C_j$ | 12 | 15 | 14 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|---|
| $e_i$ | B.V. | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | RHS | Ratio |
| 0 | $S_1$ | -1 | [1] | 0 | 1 | 0 | 0 | t | t → Key row |
| 0 | $S_2$ | -1 | 0 | 2 | 0 | 1 | 0 | t | 4 (∞) |
| 0 | $S_3$ | 1 | 1 | 1 | 0 | 0 | 1 | 100 | 100 |
| | $E_j$ | 0 | 0 | 0 | 0 | 0 | 0 | | |
| | $C_j - E_j$ | 12 | 15 | 14 | 0 | 0 | 0 | | |

↑ Key column

In the present problem we have three values in ratio column i.e. t, 4, & 100 out of these t is the least +ve. Therefore row corresponding to t in ratio column would by the key row.

Key column decides entering variable and key row decides leaving variable. Therefore in the present problem $x_2$ is entering variable i.e. it would become the basic and would enter in the solution. And $S_1$ would be the leaving variable.

★ Element at the intersection of key row & key column is the key element.

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Now all the element in the key column except the key element is to be made zero and key element is to be made unity.

Table 3.

| | $C_j$ | 12 | 15 | 14 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $C_i$ | BV | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | RHS |
| 15 | $x_2$ | -1 | 1 | 0 | 1 | 0 | 0 | t |
| 0 | $S_2$ | -1 | 0 | 2 | 0 | 1 | 0 | t |
| 0 | $S_3$ | 2 | 0 | 1 | -1 | 0 | 1 | 100-t → $R_3 \to R_3 - R_1$ |
| | $E_j$ | -15 | 15 | 0 | 15 | 0 | 0 | |
| | $C_j - E_j$ | 27 | 0 | 14 | -15 | 0 | 0 | |

In the above table $S_1$ is replaced by $x_2$ in the basic variables and corresponding $C_i$ column has also been changed.

★ Repeat the solution iteration untill the optimal solution is obtained i.e. all elements in the $C_j - E_j$ row becomes zero or -ve.

| | $C_j$ | 12 | 15 | 14 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|---|
| $C_i$ | BV | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | RHS | Ratio |
| 15 | $x_2$ | -1 | 1 | 0 | 1 | 0 | 0 | t | -t |
| 0 | $S_2$ | -1 | 0 | 2 | 0 | 1 | 0 | t | -t |
| 0 | $S_3$ | $\boxed{2}$ K.E | 0 | 1 | -1 | 0 | 1 | 100-t | $50-\frac{t}{2}$ ← key row |
| | $E_j$ | -15 | 15 | 0 | 15 | 0 | 0 | | |
| | $C_j - E_j$ | 27 | 0 | 14 | -15 | 0 | 0 | | |

↑ key column

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Now $x_1$ is entering value and $S_3$ is leaving value. Make key element unity and all the key column element other that key elment zero.

| | $C_j$ | 12 | 15 | 14 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|---|
| $C_i$ | BV | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | RHS | Ratio |
| 15 | $x_2$ | 0 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $50+\frac{t}{2}$ $(R_1 \to R_1+R_3)$ | $100+t$ |
| 0 | $S_2$ | 0 | 0 | $\left[\frac{5}{2}\right]_{K.E}$ | $-\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $50+\frac{t}{2}$ $(R_2 \to R_2+R_3)$ | $20+\frac{t}{5}$ ← key row |
| 12 | $x_1$ | 1 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $50-\frac{t}{2}$ | $100-t$ |
| | $E_j$ | 12 | 15 | $\frac{27}{2}$ | $\frac{3}{2}$ | 0 | $\frac{27}{2}$ | | |
| | $C_j-E_j$ | 0 | 0 | $\frac{1}{2}$ | $-\frac{3}{2}$ | 0 | $-\frac{27}{2}$ | | |

↑ key column

| | $C_j$ | 12 | 15 | 14 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $C_i$ | BV | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | RHS |
| 15 | $x_2$ | 0 | 1 | 0 | $\frac{3}{5}$ | $-\frac{1}{5}$ | $\frac{2}{5}$ | $40+\frac{2t}{5}$ $(R_1 \to R_1-\frac{R_2}{2})$ |
| 14 | $x_3$ | 0 | 0 | 1 | $-\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $20+\frac{t}{5}$ |
| 12 | $x_1$ | 1 | 0 | 0 | $-\frac{2}{5}$ | $-\frac{1}{5}$ | $\frac{2}{5}$ | $40-\frac{3t}{5}$ $(R_3 \to R_3-\frac{R_2}{2})$ |
| | $E_j$ | 12 | 15 | 14 | $+\frac{7}{5}$ | $\frac{1}{5}$ | $\frac{68}{5}$ | |
| | $C_j-E_j$ | 0 | 0 | 0 | $-\frac{7}{5}$ | $-\frac{1}{5}$ | $-\frac{68}{5}$ | |

It can be seen that since all the values in $C_j-E_j$ row are either −ve or zero, hence optimal solution has been reached.

Final solution $x_1 = 40-\frac{3t}{5}$ ; t is very small quantity and hence can be negleted

therefore
$$\boxed{\begin{aligned} x &= 40 \\ x_2 &= 40 \\ x_3 &= 20 \end{aligned}}$$

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