Yukawa Theory of Nuclear Force

4.4 Meson theory of nuclear forces

4.4.1 Concept of exchange of particles

The electromagnetic interaction between charged particles has been successfully explained to arise from an exchange of photons. Since photons are particles of zero rest mass, the electromagnetic interaction is of infinite range,

Taking the clue from the study of electromagnetic interaction, Yukawa proposed in the year 1935 that the nuclear force should arise from an exchange of mesons (now identified as pions) between nucleons. The mass of the exchanged particles decides the range of the force. A rough idea of the range of the nuclear force can be obtained from an application of the uncertainty principle.

The emission or absorption of a virtual meson results in an uncertainty of energy ΔE of the two-nucleon system.

$$\Delta E = m_\pi c^2 \approx 140 \text{ MeV}, \tag{4.26}$$

where m_π is the mass of the meson and c, the velocity of light. The corresponding uncertainty in time is given by

$$\Delta t = \frac{\hbar}{\Delta E} = \frac{\hbar}{m_\pi c^2}. \tag{4.27}$$

Assuming that the meson travels with a velocity approaching c, the range of the force r_0 becomes equal to the farthest distance that the meson can go in this time.

$$r_0 \approx c\,\Delta t = \frac{\hbar}{m_\pi c} = \frac{1}{\mu} \approx 1.4 \text{ fm}. \tag{4.28}$$

Thus the range of nuclear force is roughly equal to the rationalized or reduced compton wavelength of the pion $\hbar/m_\pi c$. In the case of electromagnetic interaction, the exchanged quantum is of zero mass and hence the electromagnetic interaction is of infinite range.

4.4.2 The Yukawa potential

The relativistic mass–energy relation for the pion is given by

$$E^2 = p^2c^2 + m_\pi^2 c^4. \tag{4.29}$$

Making the usual substitution of quantum mechanical operators for E and p,

$$E = i\hbar\frac{\partial}{\partial t}, \qquad p = -i\hbar\boldsymbol{\nabla}, \tag{4.30}$$

and rearranging, we obtain the field equation for the pion

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \mu^2\right)\Phi(\boldsymbol{r},t) = 0, \tag{4.31}$$

with $\mu = m_\pi c/\hbar$. This is known as the Klein–Gordon equation and it is the relativistic wave equation applicable for spin–zero particles. By setting $\mu = 0$ in Eq. (4.31), we obtain Maxwell's electromagnetic wave equation.

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi(\boldsymbol{r},t) = 0. \tag{4.32}$$

For electrostatic field $\partial\Phi/\partial t = 0$ and the wave equation (4.32) reduces to Laplace equation in free space.

$$\boldsymbol{\nabla}^2\phi(\boldsymbol{r}) = 0. \tag{4.33}$$

In the presence of a point source of charge q placed at the origin, the field equation becomes the Poisson equation

$$\boldsymbol{\nabla}^2\phi(\boldsymbol{r}) = -\frac{q}{\epsilon_0}\delta(\boldsymbol{r}), \tag{4.34}$$

the solution of which

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r}, \tag{4.35}$$

gives the scalar potential at a distance r from the point charge q placed at the origin.

To verify that (4.35) is the solution (4.34), we need the important relation

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(\boldsymbol{r}). \tag{4.36}$$

In an analogous manner, we can write the static field equation for the pion from Eq. (4.31) and obtain its solution.

$$\left(\nabla^2 - \mu^2\right) \phi(\boldsymbol{r}) = 0. \tag{4.37}$$

Equation (4.37) is the free field equation for the pions. Just as charges serve as sources for the electrostatic field, the nucleons serve as sources for the pion field. For a nucleon placed at the origin of the coordinate system, the pion field equation can be written as

$$\left(\nabla^2 - \mu^2\right) \phi(\boldsymbol{r}) = G\,\delta(\boldsymbol{r}), \tag{4.38}$$

where the constant G plays the same role as the charge in the electrostatic case. The solution of Eq. (4.38) becomes

$$\phi(\boldsymbol{r}) = -\frac{G}{4\pi}\frac{e^{-\mu r}}{r}. \tag{4.39}$$

It can be easily verified that (4.39) is the solution of the pion field equation (4.38) (*vide* problem 4.7). Now consider a second nucleon located at $\boldsymbol{R}$. The corresponding potential energy is given by

$$\begin{aligned} V &= G\int \psi^*(\boldsymbol{R})\psi(\boldsymbol{R})\phi(\boldsymbol{r})d^3r = G\int \delta(\boldsymbol{r}-\boldsymbol{R})\,\phi(\boldsymbol{r})\,d^3r \\ &= G\phi(\boldsymbol{R}) = -\frac{G^2}{4\pi}\frac{e^{-\mu R}}{R} = -g^2\frac{e^{-\mu R}}{R}, \end{aligned} \tag{4.40}$$

where $g^2 = G^2/4\pi$ and R is the inter-nucleon distance. This is the Yukawa potential for the nucleon–nucleon interaction.