# Types of Exchange Forces

## 4.2 General form of the nucleon–nucleon potential

For simplicity, let us assume that the nuclear force is derivable from a potential and it is velocity independent. Then the nucleon–nucleon potential should be a function of the three dynamical variables, the relative position coordinate, $\boldsymbol{r}$ and the spins $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ of the two nucleons, neglecting, for the moment, the isospin. Besides, it should be a scalar function in the combined space of configuration and spin since the potential should be invariant under rotations and reflections of the coordinate system. The only distinct scalar functions that can be be constructed from the space-spin coordinates are $V(r)$, $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$, $(\boldsymbol{\sigma}_1 \cdot \boldsymbol{r})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{r})$. Hence

$$V(\boldsymbol{r}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = V_r(r) + V_\sigma(r)\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + V_T(r)S_{12}, \tag{4.6}$$

where $V_r(r)$, $V_\sigma(r)$, $V_T(r)$ are functions of the scalar $r$ (the inter–nucleon distance) and they are of the Yukawa form

$$V(r) = e^{-\mu r}/\mu r, \tag{4.7}$$

where $\mu = 1/r_n$ is a constant. $S_{12}$ is a tensor operator defined so as to make its angular average zero.

$$S_{12} = \frac{3}{r^2}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{r})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{r}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2. \tag{4.8}$$

Although $S_{12}$ is a scalar in the combined space of configuration and spin, it is known as the tensor force since it can be shown to consist of two tensor operators, each of rank 2, when considered separately in the configuration and spin space (*vide* problem 4.4).

$$S_{12} = \sqrt{\frac{24\pi}{5}}\left(Y_2(\hat{r}) \cdot (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)_2\right). \tag{4.9}$$

The first two terms in expression (4.6) are spherically symmetric and hence they are known to constitute central potential. Since[3]

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = \begin{cases} +1, & S = 1 \quad \text{(triplet spin)} \\ -3, & S = 0 \quad \text{(singlet spin)} \end{cases} \tag{4.10}$$

the spin–spin interaction is such that it is attractive for triplet spin and repulsive for singlet spin. This repulsive component in the nuclear potential, besides the hard core, is responsible for the saturation properties of the nucleus.

[3] *vide* problem 4.2.

It is now simple to write down the complete nucleon–nucleon potential including their isospin $\tau_1$ and $\tau_2$.

$$\begin{aligned} V(\boldsymbol{r}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2) &= V_r(r) + V_\sigma(r)\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + V_T(r)S_{12} + V_\tau(r)\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \\ &+ V_{\sigma\tau}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) + V_{T\tau}S_{12}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2). \end{aligned} \tag{4.11}$$

Due to the presence of non-central forces ($S_{12}$ term), the two–nucleon system has only the following constants of motion

$$\boldsymbol{J}^2,\ J_z,\ P,\ \boldsymbol{S}^2,$$

where $\boldsymbol{J}$ denotes the total angular momentum operator, $J_z$, its projection, $P$ the parity operator and $\boldsymbol{S}$ the total spin. $\boldsymbol{L}^2$, $L_z$ and $S_z$ are no longer the constants of motion in the presence of non–central forces. Due to parity conservation, a state of even parity will involve a mixture of even $L$ states whereas a state of odd parity will involve a state of odd $L$ states. $\boldsymbol{S}^2$ is a constant of motion since $S_{12}$ is invariant under exchange of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$.

## 4.3 Exchange forces

Expression (4.11) has been derived above from the general considerations of the nucleon–nucleon potential but historically it has been arrived at by invoking exchange forces. To explain the saturation properties of nuclear matter, it was found necessary to postulate exchange forces besides the no–exchange Wigner central force. There are three types of exchange forces, the first arising from the spatial exchange (Majorana force), the second arising from the spin exchange (Bartlett force) and the third arising from the space–spin exchange (Heisenberg force). It is simple to construct these exchange operators which have eigenvalues $\pm 1$, depending upon the symmetry of the two-nucleon system. For instance, the spin exchange operator $P_\sigma$ can be written as

$$P_\sigma = \frac{1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{2}, \tag{4.12}$$

so that

$$P_\sigma\, {}^3\chi_m = {}^3\chi_m, \text{ symmetric triplet spin state,} \tag{4.13}$$

$$P_\sigma\, {}^1\chi_0 = -{}^1\chi_0, \text{ antisymmetric singlet spin state.} \tag{4.14}$$

The total wave function of the two–nucleon system is a product of its space, spin and isospin wave functions and by the Pauli exclusion principle, the total wave function should be antisymmetric.

$$P_r P_\sigma P_\tau = -1. \tag{4.15}$$

In the above equation, $P_r$ and $P_\tau$ denote respectively the spatial and isospin exchange operators. The isospin exchange operator can be written in analogy with the spin exchange operator (4.12).

$$P_\tau = \frac{1 + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2}{2}. \tag{4.16}$$

From Eqs. (4.12), (4.15) and (4.16), we obtain

$$P_r = -\frac{1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{2} \frac{1 + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2}{2}. \tag{4.17}$$

The concept of exchange forces thus lead to the two-nucleon potential,

$$V = V_W(r) + V_M(r)\, P_r + V_B(r)\, P_\sigma + V_H(r)\, P_r P_\sigma, \tag{4.18}$$

where $V_W(r), V_M(r), V_B(r)$ and $V_H(r)$ are of Yukawa form $e^{-\mu r}/\mu r$ with different values for the constant $\mu$. Substituting the expressions for $P_r$, $P_\sigma$ and $P_r P_\sigma (= -P_\tau)$ and rearranging, one will obtain all the terms except the tensor potential terms in Eq. (4.11). However, it is found necessary to postulate the existence of tensor force in addition to these exchange forces to explain the quadrupole moment of the deuteron.

It is instructive to see how the exchange forces are attractive in some states and repulsive in other states of the two-nucleon system. For this, let us consider the Schrödinger equation of the two-nucleon system for which the reduced mass is $M/2$, $M$ being the mass of the nucleon.

$$\left\{-\frac{\hbar^2}{M}\nabla^2 + V(r)\right\}\psi(1,2) = E\psi(1,2), \tag{4.19}$$

$$\left\{\frac{\hbar^2}{M}\nabla^2 + E\right\}\psi(1,2) = V(r)\psi(1,2), \tag{4.20}$$

where $V(r)$ is an attractive potential. Including the spin wave function, $\psi(1,2)$ can be written as a product of space and spin wave functions $\phi(1,2)\chi(1,2)$.

$$\psi(1,2) = \phi(1,2)\chi(1,2).$$

**Wigner force**

It is a no-exchange force.

$$V(r) = V_W(r).$$

It is attractive for all states of the two-nucleon system.

**Majorana force**

The Majorana force arises from spatial exchange.

$$V(r) = V_M(r)\,P_r.$$

$$P_r\phi(1,2) = \phi(2,1) = \begin{cases} \phi(1,2), & \text{for even parity states} \\ -\phi(1,2), & \text{for odd parity states.} \end{cases} \tag{4.21}$$

Thus the Majorana exchange force is attractive for even parity states and repulsive for odd parity states.

**Bartlett force**

The Bartlett force arises from spin-exchange.

$$V(r) = V_B(r)\,P_\sigma.$$

$$\left\{\frac{\hbar^2}{M}\nabla^2 + E\right\}\phi(1,2)\chi(1,2) = V_B(r)\,P_\sigma\,\phi(1,2)\chi(1,2)$$

$$P_\sigma\chi(1,2) = \chi(2,1) = \begin{cases} +\chi(1,2), & \text{triplet spin state} \\ -\chi(1,2), & \text{singlet spin state.} \end{cases} \tag{4.22}$$

Thus we find the Bartlett exchange force is attractive for triplet spin state and repulsive for singlet spin state.

**Heisenberg force**

The Heisenberg force arises from space-spin exchange.

$$V(r) = V_H(r)\,P_r P_\sigma.$$

$$\left\{\frac{\hbar^2}{M}\nabla^2 + E\right\}\phi(1,2)\chi(1,2) = V_H(r)P_rP_\sigma\,\phi(1,2)\chi(1,2).$$

$$\begin{aligned} P_rP_\sigma\phi(1,2)\chi(1,2) &= \phi(2,1)\chi(2,1) \\ &= \begin{cases} +\phi(1,2)\chi(1,2) & \text{even-parity triplet-spin state or odd-parity spin-singlet state} \\ -\phi(1,2)\chi(1,2) & \text{odd-parity triplet-spin state or even-parity spin-singlet state.} \end{cases} \end{aligned} \tag{4.23}$$

Thus the Heisenberg force is attractive for even-parity spin-triplet state and odd-parity singlet-spin state but repulsive for odd-parity triplet-spin state and even-parity singlet-spin state.