**MA/M.Sc. (SEMESTER-IV)**
**(Orthogonal Polynomials)**

## Determinacy of £ in the Bounded Case

### Definition

A polynomial $q(x)$, not identically zero, is called *quasi-orthogonal polynomial of order n+1* if and only if it is of degree at most $n+1$ and

$$£\left[x^k q(x)\right] = 0 \qquad \text{for} \quad k = 0,1,........n-1.$$

Note that according to this definition $P_n(x)$ and $P_{n+1}(x)$ are both quasi-orthogonal polynomial of order $n+1$.

### Theorem

(i) $q(x)$ is a quasi-orthogonal polynomial of order $n+1$ if and only if there are constants *A* and *B*, not both zero, such that

$$q(x) = AP_{n+1}(x) + BP_n(x)$$

(ii) For each number $z_0$, there is a quasi-orthogonal polynomial of order $n+1$, $q(x)$, such that $q(z_0) = 0$. This $q(x)$ is uniquely determined up to an arbitrary non-zero factor, and its degree is n+1 if and only if $P_n(z_0) \neq 0$.

### Proof

Let

$$q(x) = AP_{n+1}(x) + BP_n(x)$$

then

$$£\left[x^k q(x)\right] = £\left[Ax^k P_{n+1}(x) + Bx^k P_n(x)\right], \qquad k = 0,1,........n-1$$

$$= A£\left[x^k P_{n+1}(x)\right] + B£\left[x^k P_{n+1}(x)\right]$$

$$= 0 \qquad \text{if} \quad |A| + |B| \neq 0$$

Hence $q(x)$ is a quasi-orthogonal polynomial of order $n+1$.

Conversely, let $q(x)$ is a quasi-orthogonal polynomial of order $n+1$, so we can write

$$q(x) = \sum_{k=0}^{n+1} c_k P_k(x)$$

where $c_k = \{£[P_k^2(x)]\}^{-1} £[q(x)P_k(x)] = 0 \quad \text{for} \quad 0 \le k \le n-1$.

Hence

$$q(x) = AP_{n+1}(x) + BP_n(x).$$

(ii) Let $z_0$ be a zero of $P_n(x)$ or $P_{n+1}(x)$. Now

$$q(x) = AP_{n+1}(x) + BP_n(x) \qquad \text{where } A \text{ and } B \text{ are not both zero.}$$

If $P_n(z_0) = 0$ choose constant $A = 0,\ B \ne 0.$

Then $q(z_0) = 0$.

Similarly for if $P_{n+1}(z_0) = 0$.

If $P_n(z_0) \ne 0 \Rightarrow B = 0,\ A \ne 0.$

So $q(x) = AP_{n+1}(x)$

Hence $q(x)$ is a polynomial of degree $n+1$.
Since $q(z_0) = 0 \Rightarrow P_{n+1}(z_0) = 0$. So $z_0$ is a zero of $P_{n+1}(x)$
Hence it cannot be a zero $P_n(x)$ i.e. $P_n(z_0) \ne 0.$

**Theorem**

The zeros of a real quasi-orthogonal are all real and simple. At most one of these lies outside the open interval, $(\xi_1, \eta_1)$.

**Proof**

If $q(x)$ is an orthogonal polynomial there is nothing to prove. Let $q(x) = AP_{n+1}(x) + BP_n(x)$ where A and B are real and different from zero.

Let $x_{n+1,i}$ , $i = 1,.......,n+1$, be the zeros of $P_{n+1}(x)$. Then

$$q(x_{n+1,i}) = BP_n(x_{n+1,i})$$

As $i$ will vary from $i = 1,.......,n+1$ $P_n(x)$ will change sign, so $q(x)$ will change sign as $i$ varies from 1 to n+1. So $q(x)$ has n real zeros separating the n+1 zeros of $P_{n+1}(x)$ . Since $q(x)$ is real, its remaining one zero must be real and must lie outside $\lfloor x_{n+1,1}, x_{n+1,n+1} \rfloor$.

**Theorem**

Let $x_0$ be any real number which is not a zero of $P_n(x)$. Let $q(x)$ denote a real quasi-orthogonal polynomial of order and degree n+1 which vanishes at $x_0$. If $\Lambda_{n0}$ denotes the quadrature coefficient which corresponds to $\pi(x_0) = 1$.

$$\Lambda_{\mathrm{n0}} = \min £ \left[ | \pi(x) |^2 \right]$$

where the minimum is computed as $\pi(x)$ ranges over all polynomials of degree at most n such that $\pi(x_0) = 1$.

**Proof**

Suppose $y_{np} = x_0$ and write $B_{np} = \Lambda_{\mathrm{n0}}$. If $\pi(x)$ has degree not exceeding n and $\pi(x_0) = 1$, then

$$£\left[| \pi(x) |^2\right] \geq B_{np} \, | \pi(x_0) |^2 = \Lambda_{n0} .$$

Hence $\Lambda_{\mathrm{n0}} = \min £ \left[ | \pi(x) |^2 \right]$.

Now the polynomial

$$\rho(x) = \frac{q(x)}{(x - x_0) q^{'}(x_0)}$$

is of degree n and $\rho(x)$ will vanish at $y_{ni}$ for $i \neq p$ and $\rho(x_0) = 1$.

That is,

$$£\left[\rho^2(x)\right] = \Lambda_{\mathrm{n0}} .$$

**Corollary**

$$\Lambda_{\mathrm{n0}} = \left\{ \sum_{k=0}^{n} p_k^2(x_0) \right\}^{-1}$$

where $p_k(x)$ denotes the kth orthonormal polynomial.

**Proof**

$$\Lambda_{\mathrm{n}0} = [K_n(x_0, x_0)]^{-1} = \left\{\sum_{k=0}^{n} \overline{p_k(x_0)} p_k(x_0)\right\}^{-1}$$

$$= \left\{\sum_{k=0}^{n} p_k \overline{(x_0)} p_k(x_0)\right\}^{-1}$$

$$= \left\{\sum_{k=0}^{n} p_k^2(x_0)\right\}^{-1}.$$

**Theorem**

Let $\phi$ be any representative of £ . The for any real number $x_0$,

$$\phi(x_0) - \phi(-\infty) \leq \left\{\sum_{k=0}^{\infty} p_k^2(x_0)\right\}^{-1} \qquad \text{if } -\infty < x_0 \leq \xi_1$$

$$\phi(+\infty) - \phi(x_0) \leq \left\{\sum_{k=0}^{\infty} p_k^2(x_0)\right\}^{-1} \qquad \text{if } \eta_1 < x_0 \leq +\infty$$

**Proof**

Suppose that $-\infty < x_0 \leq \xi_1$ and let $q(x)$ be a real quasi-orthogonal polynomial of order and degree n+1 that has $x_0$ as a zero. Now

$$\Lambda_{\mathrm{n}0} = £\left[\rho^2(x)\right] = \int_{-\infty}^{\infty} \rho^2(x) d\varphi(x) \geq \int_{-\infty}^{x_0} \rho^2(x) d\varphi(x)$$

Since $q(x)$ has no zero smaller than $\xi_1$ other than $x_0$. It follows that

$$\rho^2(x) > \rho^2(x_0) = 1 \text{ for } x < x_0,$$

hence

$$\Lambda_{\mathrm{n}0} \geq \int_{-\infty}^{x_0} d\varphi(x) = \varphi(x_0) - \varphi(-\infty).$$

Since $\Lambda_{n0} = \left\{ \sum_{k=0}^{n} p_k^2(x_0) \right\}^{-1}$, we have

$$\varphi(x_0) - \varphi(-\infty) \le \left\{ \sum_{k=0}^{n} p_k^2(x_0) \right\}^{-1} \quad \text{for} \quad x_0 \le \xi_1.$$

The remaining case is proved in the same way.

**Theorem**

Let $[a,b]$ be a compact interval and let $\phi_1$ and $\phi_2$ be functions of bounded variation on $[a,b]$ such that

$$\int_a^b x^n d\phi_1(x) = \int_a^b x^n d\phi_2(x), \qquad \text{n} = 0, 1, 2, \ldots.$$

Then there exist a constant C such that $\phi_1(x) - \phi_2(x) = C$ at all $x \in [a,b]$ at which both are continuous.

**Proof**

Let

$$\phi(x) = \phi_1(x) - \phi_2(x)$$

So that $\phi$ is of bounded variation on [a,b] and

$$\int_a^b x^n d\phi(x) = \int_a^b x^n d\phi_1(x) - \int_a^b x^n d\phi_2(x) = 0 \ .$$

Hence

$$\int_a^b \pi(x) d\phi(x) = 0 \quad \text{for every polynomial } \pi(x).$$

Now if $f$ is any continuous function on [a,b], then by Weierstrass approximation theorem $f$ can be approximated by polynomials. Hence we have

$$\int_a^b f(x)d\phi(x) = 0 \ .$$

Taking $f(x) = 1$, we have $\phi(b) - \phi(a) = 0$.

Now for any $t$, $a < t < b$ which is point of continuity of $\phi$, define

$$f(x) = \begin{cases} x & a \le x \le t \\ t & t < x \le b \end{cases}$$

Then $f$ is continuous on [a,b] and we have

$$0 = \int_a^b f d\phi(x) = \int_a^t x d\phi(x) + \int_t^b t d\phi(x)$$

$$= t\phi(t) - a\phi(a) - \int_a^t \phi(x)dx + t\phi(b) - t\phi(t)$$

$$= (t-a)\phi(a) - \int_a^t \phi(x)dx \ .$$

Hence

$$\Phi(t) = \int_a^t \phi(x)dx = (t-a)\phi(a)$$

Since is $\phi$ continuous at $t$, $\Phi^{'}(t)$ exist and we have

$$\Phi^{'}(t) = \phi(t) = \phi(a)$$

so

$$\phi_1(t) - \phi_2(t) = \phi(a) .$$

Note : These notes were taught and given to the students in the class.