Chapter 8
Laplace Transforms and Inverse Laplace Transforms

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Definition of the Laplace Transform

Let $f(x)$ be defined for $0 \leq x < \infty$ and let s denote an arbitrary real variable. The *Laplace transform of* $f(x)$, designated by either $\mathcal{L}\{f(x)\}$ or $F(s)$, is

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)dx \tag{8.1}$$

for all values of s for which the improper integral converges. Convergence occurs when the limit

$$\lim_{R\to\infty} \int_0^R e^{-sx} f(x)dx \tag{8.2}$$

exists. If this limit does not exist, the improper integral diverges and $f(x)$ has no Laplace transform. When evaluating the integral in Equation 8.1, the variable s is treated as a constant because the integration is with respect to x.

The Laplace transforms for a number of elementary functions are given in Appendix A.

Properties of Laplace Transforms

Property 8.1 (Linearity). If $\mathcal{L}\{f(x)\} = F(s)$ and $\mathcal{L}\{g(x)\} = G(s)$, then for any two constants c_1 and c_2

$$\begin{aligned}\mathcal{L}\{c_1 f(x) + c_2 g(x)\} &= c_1\mathcal{L}\{f(x)\} + c_2\mathcal{L}\{g(x)\} \\ &= c_1 F(s) + c_2 G(s)\end{aligned} \tag{8.3}$$

Property 8.2. If $\mathcal{L}\{f(x)\} = F(s)$, then for any constant a

$$\mathcal{L}\{e^{ax}f(x)\} = F(s-a) \tag{8.4}$$

Property 8.3. If $\mathcal{L}\{f(x)\} = F(s)$, then for any positive integer n

$$\mathcal{L}\{x^n f(x)\} = (-1)^n \frac{d^n}{ds^n}[F(s)] \tag{8.5}$$

Property 8.4. If $\mathcal{L}\{f(x)\} = F(s)$ and if $\lim\limits_{\substack{x\to 0\\ x>0}} \frac{f(x)}{x}$ exists, then

$$\mathcal{L}\left\{\frac{1}{x}f(x)\right\} = \int_s^\infty F(t)dt \tag{8.6}$$

Property 8.5. If $\mathcal{L}\{f(x)\} = F(s)$, then

$$\mathcal{L}\left\{\int_0^x f(t)dt\right\} = \frac{1}{s}F(s) \tag{8.7}$$

Property 8.6. If $f(x)$ is periodic with period ω, that is, $f(x+\omega) = f(x)$, then

$$\mathcal{L}\{f(x)\} = \frac{\int_0^\omega e^{-sx}f(x)dx}{1-e^{-\omega s}} \tag{8.8}$$

Functions of Other Independent Variables

For consistency only, the definition of the Laplace transform and its properties, Equations 8.1 through 8.8, are presented for functions of x. They are equally applicable for functions of any independent variable and are generated by replacing the variable x in the above equations by any variable of interest. In particular, the counter part of Equation 8.1 for the Laplace transform of a function of t is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt$$

Definition of the Inverse Laplace Transform

An inverse *Laplace transform* of $F(s)$ designated by $\mathcal{L}^{-1}\{F(s)\}$, is another function $f(x)$ having the property that $\mathcal{L}\{f(x)\} = F(s)$.

The simplest technique for identifying inverse Laplace transforms is to recognize them, either from memory or from a table such as in the Appendix. If $F(s)$ is not in a recognizable form, then occasionally it can be transformed into such a form by algebraic manipulation. Observe from the Appendix that almost all Laplace transforms are quotients. The recommended procedure is to first convert the denominator to a form that appears in the Appendix and then the numerator.

Manipulating Denominators

The method of *completing the square* converts a quadratic polynomial into the sum of squares, a form that appears in many of the denominators in the Appendix. In particular, for the quadratic

$$\begin{aligned} as^2 + bs + c &= a\left(s^2 + \frac{b}{a}s\right) + c \\ &= a\left[s^2 + \frac{b}{a}s + \left(\frac{b}{2a}\right)^2\right] + \left[c - \frac{b^2}{4a}\right] \\ &= a\left(s + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \\ &= a(s+k)^2 + h^2 \end{aligned}$$

where $k = b/2a$ and $h = \sqrt{c - (b^2/4a)}$.

The method of *partial fractions* transforms a function of the form $a(s)/b(s)$, where both $a(s)$ and $b(s)$ are polynomials in s, into the sum of other fractions such that the denominator of each new fraction is either a first-degree or a quadratic polynomial raised to some power. The method requires only that the degree of $a(s)$ be less than the degree of $b(s)$ (if this is not the case, first perform long division, and consider the remainder term) and $b(s)$ be factored into the product of distinct linear and quadratic polynomials raised to various powers.

The method is carried out as follows. To each factor of $b(s)$ of the form $(s - a)^m$, assign a sum of m fractions, of the form

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_m}{(s-a)^m}$$

To each factor of $b(s)$ of the form $(s^2 + bs + c)^p$, assign a sum of p fractions, of the form

$$\frac{B_1 s + C_1}{s^2 + bs + c} + \frac{B_2 s + C_2}{(s^2 + bs + c)^2} + \cdots + \frac{B_p s + C_p}{(s^2 + bs + c)^p}$$

Here A_i, B_j, and C_k ($i = 1,2,\ldots, m$; $j, k = 1,2,\ldots, p$) are constants which still must be determined.

Set the original fraction $a(s)/b(s)$ equal to the sum of the new fractions just constructed. Clear the resulting equation of fractions and then equate coefficients of like powers of s, thereby obtaining a set of simultaneous linear equations in the unknown constants A_i, B_j, and C_k. Finally, solve these equations for A_i, B_j, and C_k.

Manipulating Numerators

A factor $s - a$ in the numerator may be written in terms of the factor $s - b$, where both a and b are constants, through the identity $s - a = (s - b) + (b - a)$. The multiplicative constant a in the numerator may be written explicitly in terms of the multiplicative constant b through the identity

$$a = \frac{a}{b}(b)$$

Both identities generate recognizable inverse Laplace transforms when they are combined with:

Property 8.7 (Linearity). If the inverse Laplace transforms of two functions $F(s)$ and $G(s)$ exist, then for any constants c_1 and c_2,

$$\mathcal{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\,\mathcal{L}^{-1}\{F(s)\} + c_2\,\mathcal{L}^{-1}\{G(s)\}$$

Convolutions

The *convolution* of two functions $f(x)$ and $g(x)$ is

$$f(x) * g(x) = \int_0^x f(t)g(x-t)\,dt \tag{8.9}$$

Theorem 8.1. $f(x) * g(x) = g(x) * f(x)$.

Theorem 8.2. (***Convolution Theorem***). If $\mathcal{L}\{f(x)\} = F(s)$ and $\mathcal{L}\{g(x)\} = G(s)$, then $\mathcal{L}\{f(x) * g(x)\} = \mathcal{L}\{f(x)\}\,\mathcal{L}\{g(x)\} = F(s)G(s)$

You Need to Know ✓

The inverse Laplace transform of a product is computed using a convolution.

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(x) * g(x) = g(x) * f(x) \qquad (8.10)$$

If one of the two convolutions in Equation 8.10 is simpler to calculate, then that convolution is chosen when determining the inverse Laplace transform of a product.

Unit Step Function

The *unit step function* $u(x)$ is defined as

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

As an immediate consequence of the definition, we have for any number c,

$$u(x-c) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

The graph of $u(x - c)$ is given in Figure 8-1.

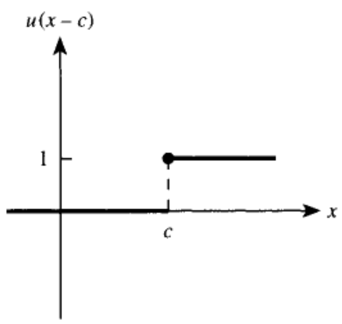

Figure 8-1

Theorem 8.3. $\mathscr{L}\{u(x-c)\} = \frac{1}{s}e^{-cs}$.

Translations

Given a function $f(x)$ defined for $x \geq 0$, the function

$$u(x-c)f(x-c) = \begin{cases} 0 & x < c \\ f(x-c) & x \geq c \end{cases}$$

represents a shift, or translation, of the function $f(x)$ by c units in the positive x-direction. For example, if $f(x)$ is given graphically by Figure 8-2, then $u(x-c)f(x-c)$ is given graphically by Figure 8-3.

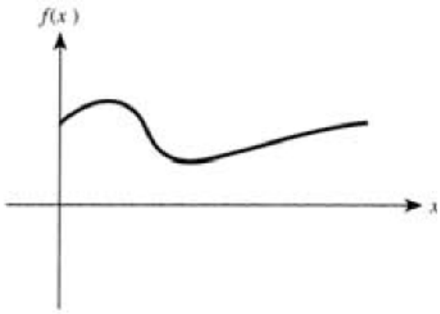

Figure 8-2

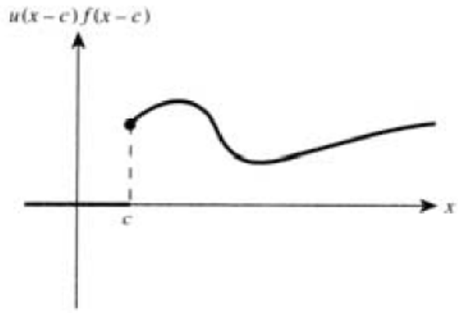

Figure 8-3

Theorem 8.4. If $F(s) = \mathscr{L}\{f(x)\}$, then

$$\mathscr{L}\{u(x-c)f(x-c)\} = e^{-cs}F(s)$$

Conversely,

$$\mathscr{L}^{-1}\{e^{-cs}F(s)\} = u(x-c)f(x-c) = \begin{cases} 0 & x < c \\ f(x-c) & x \geq c \end{cases}$$

Solved Problems

Solved Problem 8.1 Find $\mathscr{L}\{e^{ax}\}$.

Using Equation 8.1, we obtain

$$\begin{aligned} F(s) &= \int_0^\infty e^{-sx}e^{ax}dx = \lim_{R\to\infty}\int_0^R e^{(a-s)x}dx \\ &= \lim_{R\to\infty}\left[\frac{e^{(a-s)x}}{a-s}\right]_{x=0}^{x=R} = \lim_{R\to\infty}\left[\frac{e^{(a-s)R}-1}{a-s}\right] \\ &= \frac{1}{s-a} (\text{for } s > a) \end{aligned}$$

Note that when $s \leq a$, the improper integral diverges. (See also entry 7 in the Appendix.)

Solved Problem 8.2 Find $\mathscr{L}\{xe^{4x}\}$.

This problem can be done three ways.

(a) Using entry 14 of the Appendix with $n = 2$ and $a = 4$, we have directly that

$$\mathscr{L}\{xe^{4x}\} = \frac{1}{(s-4)^2}$$

(b) Set $f(x) = x$. Using Property 8.2 with $a = 4$ and entry 2 of the Appendix, we have

$$F(s) = \mathscr{L}\{f(x)\} = \mathscr{L}\{x\} = \frac{1}{s^2}$$

and

$$\mathscr{L}\{e^{4x}x\} = F(s-4) = \frac{1}{(s-4)^2}$$

(c) Set $f(x) = e^{4x}$. Using Property 8.3 with $n = 1$ and the results of Problem 8.1, or alternatively, entry 7 of the Appendix with $a = 4$ we find that

$$F(s) = \mathscr{L}\{f(x)\} = \mathscr{L}\{e^{4x}\} = \frac{1}{s-4}$$

and

$$\mathscr{L}\{xe^{4x}\} = -F'(s) = -\frac{d}{ds}\left(\frac{1}{s-4}\right) = \frac{1}{(s-4)^2}$$

Solved Problem 8.3 Use partial fractions to decompose $\dfrac{s+3}{(s-2)(s+1)}$.

To the linear factors $s - 2$ and $s + 1$, we associate respectively the fractions $A/(s - 2)$ and $B/(s + 1)$. We set

$$\frac{s+3}{(s-2)(s+1)} \equiv \frac{A}{s-2} + \frac{B}{s+1}$$

and, upon clearing fractions, obtain

$$s+3 \equiv A(s+1) + B(s-2) \tag{8.11}$$

To find A and B, we substitute $s = -1$ and $s = 2$ into 8.11, we immediately obtain $A = 5/3$ and $B = -2/3$. Thus,

$$\frac{s+3}{(s-2)(s+1)} \equiv \frac{5/3}{s-2} - \frac{2/3}{s+1}$$

Solved Problem 8.4 Find $\mathscr{L}^{-1}\left\{\dfrac{s+3}{(s-2)(s+1)}\right\}$.

No function of this form appears in the Appendix. Using the results of Problem 8.3 and Property 8.7, we obtain

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s+3}{(s-2)(s+1)}\right\} &= \frac{5}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{2}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= \frac{5}{3}e^{2x} - \frac{2}{3}e^{-x}\end{aligned}$$

Chapter 9
SOLUTIONS BY LAPLACE TRANSFORMS

IN THIS CHAPTER:

- ✔ *Laplace Transforms of Derivatives*
- ✔ *Solutions of Linear Differential Equations with Constant Coefficients*
- ✔ *Solutions of Linear Systems*
- ✔ *Solved Problems*

Laplace Transforms of Derivatives

Denote $\mathscr{L}\{y(x)\}$ by $Y(s)$. Then under very broad conditions, the Laplace transform of the nth-derivative ($n = 1,2,3,...$) of $y(x)$ is

$$\mathscr{L}\left\{\frac{d^n y}{dx^n}\right\} = s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0) \tag{9.1}$$

If the initial conditions on $y(x)$ at $x = 0$ are given by

$$y(0) = c_0, y'(0) = c_1, \ldots, y^{(n-1)}(0) = c_{n-1} \tag{9.2}$$

then (9.1) can be rewritten as

$$\mathscr{L}\left\{\frac{d^n y}{dx^n}\right\} = s^n Y(s) - c_0 s^{n-1} - c_1 s^{n-2} - \cdots - c_{n-2} s - c_{n-1} \tag{9.3}$$

For the special cases of $n = 1$ and $n = 2$, Equation 9.3 simplifies to

$$\mathscr{L}\{y'(x)\} = sY(s) - c_0 \tag{9.4}$$

$$\mathscr{L}\{y''(x)\} = s^2 Y(s) - c_0 s - c_1 \tag{9.5}$$

Note!

Laplace transforms convert differential equations into algebraic equations.

Solutions of Linear Differential Equations with Constant Coefficients

Laplace transforms are used to solve initial-value problems given by the nth-order linear differential equation with constant coefficients

$$b_n \frac{d^n y}{dx^n} + b_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + b_1 \frac{dy}{dx} + b_0 y = g(x) \tag{9.6}$$

together with the initial conditions specified in Equation 9.2. First, take the Laplace transform of both sides of Equation 9.6, thereby obtaining an algebraic equation for Y(s). Then solve for Y(s) *algebraically*, and finally take inverse Laplace transforms to obtain $y(x) = \mathscr{L}^{-1}\{Y(s)\}$.

Unlike previous methods, where first the differential equation is solved and then the initial conditions are applied to evaluate the arbitrary constants, the Laplace transform method solves the entire initial-value problem in one step. There are two exceptions: when no initial conditions are specified and when the initial conditions are not at $x = 0$. In these situations, c_0 through c_{n-1} in Equations 9.2 and 9.3 remain arbitrary and the solution to differential equation 9.6 is found in terms of these constants. They are then evaluated separately when appropriate subsidiary conditions are provided.

Solutions of Linear Systems

Laplace transforms are useful for solving systems of linear differential equations; that is, sets of two or more differential equations with an equal number of unknown functions. If all of the coefficients are constants, then the method of solution is a straightforward generalization of the one described above. Laplace transforms are taken of each differential equation in the system; the transforms of the unknown functions are determined algebraically from the resulting set of simultaneous equations; inverse transforms for the unknown functions are calculated with the help of the Appendix.

Solved Problems

Solved Problem 9.1 Solve $y' - 5y = e^{5x}$; $y(0) = 0$.

Taking the Laplace transform of both sides of this differential equation and using Property 8.1, we find that $\mathscr{L}\{y'\} - 5\mathscr{L}\{y\} = \mathscr{L}\{e^{5x}\}$. Then, using the Appendix and Equation 9.4 with $c_0 = 0$, we obtain

$$[sY(s) - 0] - 5Y(s) = \frac{1}{s-5} \text{ from which } Y(s) = \frac{1}{(s-5)^2}$$

Finally, taking the inverse Laplace transform of $Y(s)$, we obtain

$$y(x) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{(s-5)^2}\right\} = xe^{5x}$$

(see Appendix, entry 14).

Solved Problem 9.2 Solve the system

$$\begin{aligned} y'' + z + y &= 0 \\ z' + y' &= 0; \\ y(0) = 0, \quad y'(0) &= 0, \quad z(0) = 1 \end{aligned}$$

Denote $\mathscr{L}\{y(x)\}$ and $\mathscr{L}\{z(x)\}$ by $Y(s)$ and $Z(s)$ respectively. Then, taking the Laplace transforms of both differential equations, we obtain

$$\begin{aligned}[s^2Y(s)-(0)s-(0)]+Z(s)+Y(s)&=0\\ [sZ(s)-1]+[sY(s)-0]&=0\end{aligned}$$

or

$$\begin{aligned}(s^2+1)Y(s)+Z(s)&=0\\ Y(s)+Z(s)&=\frac{1}{s}\end{aligned}$$

Solving this last system for $Y(s)$ and $Z(s)$, we find that

$$Y(s)=-\frac{1}{s^3} \qquad Z(s)=\frac{1}{s}+\frac{1}{s^3}$$

Thus, taking inverse transforms, we conclude that

$$y(x)=-\frac{1}{2}x^2 \quad z(x)=1+\frac{1}{2}x^2$$