P-1

Unit-III

Lie Group:

Let $G \to$ grp, which is also a diff. Manifold.
Let ϕ_1, ϕ_2 be maps given by

$$\phi_1 : G \times G \to G \ \& \ \phi_2 : G \to G$$

defined by

$$\phi_1(g_1, g_2) = g_1 g_2 \ \& \ \phi_2(g) = g^{-1}$$

$$\forall g, g_1, g_2 \in G.$$

If ϕ_1, ϕ_2 are both are differentiable, Then G is ~~called~~ a Lie Group.

Examples

Question:

(1) ~~Show that~~ The set $R^+ = \{x \in R \mid x > 0\}$ is a Lie grp with respect to mult^n.

Pf. $G \equiv R^+$

Let $x, y \in R^+$

Let $\phi_1 : R^+ \times R^+ \to R^+$ defined by $\phi_1(x, y) = xy$ $\forall x, y \in R^+$

& $\phi_2 : R^+ \to R^+$, defined by $\phi_2(x) = \frac{1}{x}$, $\forall x \in R^+$

Obviously ϕ_1, ϕ_2 are diff ble, $\Rightarrow R^+$ is Lie grp.

(2) R is a Lie grp wrt addition.

Let $G \equiv R$

Let $x, y \in R$

Define $\phi_1 : R \times R \to R$

s.t $\phi_1(x, y) = x + y$

& $\phi_2 : R \to R$,

s.t. $\phi_2(x) = -x$

$\forall x, y \in R$

Obviously ϕ_1, ϕ_2 are diff $\Rightarrow$ R is Lie grp.

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P-2.

$Gl(n,\mathbb{R}) \equiv \mathbb{R}^{n^2}$ is also a Lie ssp w.r.t matx mult.

Set of all non singular Matx.

$\rightarrow \phi_1 : Gl(n,\mathbb{R}) \times Gl(n,\mathbb{R}) \longrightarrow Gl(n,\mathbb{R})$

$\phi_1(A,B) = A \cdot B$ & $\phi_2 : Gl(n,\mathbb{R}) \to Gl(n,\mathbb{R})$, $\phi_2(A) = A^{-1}$ $\forall A, B \in Gl(n,\mathbb{R})$ where $A = [a_{ij}]$, $B = [b_{ij}]$

As ϕ_1, ϕ_2 are diff ble $\Rightarrow$ $Gl(n,\mathbb{R})$ is Lie ssp.

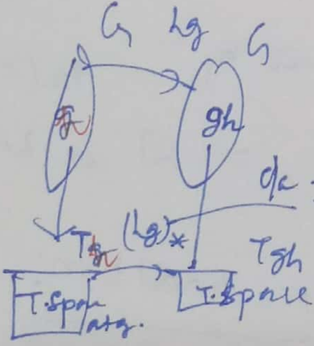
Left Translation & Right translation:

Let $G \to$ Lie ssp. & $g \in G$.

Then The mapping $L_g : G \to G$ defined as $L_g(h) = gh$, $\forall h \in G$, is called Left translation.

Similarly. The right translation $R_g : G \to G$ is defined $R_g(h) = hg$, $\forall h \in G$.

Note: (1)



called Jacobian of L_g.

~~$(L_g)_* : T_g \to T_{gh}$ is defined by~~

Qs If $L_g \circ R_{g^{-1}} = I_g$, find $I_g(h) = ?$

$\because I_g(h) = (L_g \circ R_{g^{-1}})(h)$
$= L_g(R_{g^{-1}}(h))$
$= L_g(hg^{-1}) = ghg^{-1}$

P-3.

Lie Algebra

Lie Algebra over $\mathbb{R}$ is a Real vector Sp V, equipped with bilinear map $[\cdot,\cdot] : V \times V \to V$ satisfying —

① $[x,x] = 0$

② $[x,y] = -[y,x]$

③ $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$

Thus, The Space of vector fields on M, equipped with ~~Lie bracket~~ $[\cdot,\cdot]$ as a bin opⁿ, is c/a a Lie Algebra..

i.e $(T_m M, [\cdot,\cdot]) \to$ Lie Algebra.

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Homomorphism & Isomorphism:

Let $G_1, G_2 \to$ Lie grp.

If diff map $\phi : G_1 \to G_2$ satisfying $\phi(gh) = \phi(g) \cdot \phi(h)$, $\forall g,h \in G_1$

Then ϕ is c/a Homomorphism.

Further If ϕ is a bijective Homo, Then ϕ is c/a Isomorphism from G_1 into G_2.

Note $\phi[x,y] = [\phi x, \phi y]$, (if ϕ is Homo)

Defⁿ: Left Invariant vector field: A vector field X, on a Lie grp G, is c/a Left Invariant field if

$$(L_g)_* X = X,$$

07/2010
Quest: The Set of all left Invariant vector field over a Lie grp form a Lie algebra over G.

Pf: Let $V \equiv$ Set of all vector fields over Lie grp G.
$\Rightarrow V(\mathbb{R}) \to$ vector sp over field of real Nos.

Let $\mathcal{G} \equiv$ Set of all left Inv. vector fields over G.

Then $\mathcal{G} \subset V$

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Let $X, Y \in \mathcal{G}$, $a, b \in R$

$\Rightarrow$ X, Y are left Inv. field over G.

$\Rightarrow$ $(L_g)_* X = X$ & $(L_g)_* Y = Y$

Consider $(L_g)_* (\alpha X + \beta Y) = (L_{g*})(\alpha X) + (L_g)_* (\beta Y)$

$= \alpha (L_{g*}) X + \beta (L_{g*}) Y$

$= \alpha X + \beta Y$, $\forall \alpha, \beta \in R$

$\Rightarrow$ $\alpha X + \beta Y$ is ~~left~~ Inv

$\Rightarrow$ $\alpha X + \beta Y \in \mathcal{G}$.

$\Rightarrow$ $\mathcal{G}$ is Sub Sp of V.

$\Rightarrow$ $\mathcal{G}$ is vector Sp. over R.

TPT: $\mathcal{G}$ is Lie algebra over G.

For,

Let $X, Y \in \mathcal{G}$ –

$\Rightarrow$ $(L_g)_* X = X$, $(L_g)_* Y = Y$

Now $(L_{g*})[X, Y] = [(L_g)_* X, (L_g)_* Y]$

$= [X, Y]$

$\Rightarrow$ $[X, Y] \in \mathcal{G}$.

We know Lie bracket satisfy.

Properties ① $[X, X] = 0$

② $[X, Y] = -[Y, X]$

& ③ $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

Therefore $\mathcal{G}$ is Lie Algebra.

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Basis of Lie Algebra

Let $G \rightarrow$ Lie grp

& $\mathcal{G} \rightarrow$ Lie algebra of left Inv vector fields over G.

Let $B = \{x_1, x_2, \cdots x_n\}$ be a basis of Lie algebra $\mathcal{G}$.

Let $x_i, x_j \in \mathcal{G} \Rightarrow [x_i, x_j] \in \mathcal{G}$.

Thus $[x_i, x_j] = c_{ij}^1 x_1 + c_{ij}^2 x_2 + c_{ij}^3 x_3 + \cdots c_{ij}^n x_n$

$= \sum c_{ij}^r x_r$

$= c_{ij}^r x_r$.

① $[x_i, x_i] = 0. \Rightarrow c_{ii}^r x_r = 0$

$\Rightarrow \boxed{c_{ii}^r = 0}$ $\because x_r \neq 0$

$\forall i$

② $[x_i, x_j] = -[x_j, x_i]$

$\Rightarrow c_{ij}^r x_r = - c_{ji}^r x_r \Rightarrow c_{ij}^r = - c_{ji}^r$, $\because x_r \neq 0$

③ For basis vector x_i, x_j, x_k, we have

$[[x_i, x_j], x_k] + [[x_j, x_k], x_i] + [[x_k, x_i], x_j] = 0$

$\Rightarrow [c_{ij}^r x_r, x_k] + [c_{jk}^r x_r, x_i] + [c_{ki}^r x_r, x_j] = 0$

$\Rightarrow c_{ij}^r [x_r, x_k] + c_{jk}^r [x_r, x_i] + c_{ki}^r [x_r, x_j] = 0$

$\Rightarrow c_{ij}^r c_{rk}^s x_s + c_{jk}^r c_{ri}^s x_s + c_{ki}^r c_{rj}^s x_s = 0$

$\Rightarrow \boxed{c_{ij}^r c_{rk}^s + c_{jk}^r c_{ri}^s + c_{ki}^r c_{rj}^s = 0}$, $\because x_s \neq 0$ & $\{x_1, x_2, \cdots x_n\}$ is L.I.

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P-6 n1,2,07,09

Th: If the Lie grp G is of the dimension n, then the Lie algebra $\mathcal{G}$ over G is also of dim n.

Proof: Let $G \to$ Lie grp & $\dim(G) = n$ (Given)
& $\mathcal{G} \to$ Lie algebra over G.

Let $e \in G \Rightarrow T_e \equiv$ Tang. Sp. at pt e
i.e. $\dim(T_e) = n$.

TPT: $\dim(\mathcal{G}) = n$?

For,
Define a linear map.

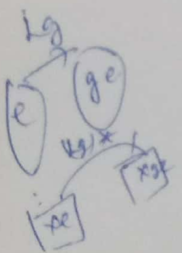
$$i_L : \mathcal{G} \to T_e \text{ s.t. } \boxed{i_L(x) = X_e}, \ \forall x \in \mathcal{G}$$

Claim: i_L is bijective homomorph i.e. $\mathcal{L} \cong T_e$.

For,

① i_L is 1-1:

If $i_L(x) = i_L(y)$
$= X_e = Y_e$



$\Rightarrow (L_g)_* X_e = (L_g)_* Y_e$ where $g \in G$, $L_g \to$ left translation, $(L_g)_* \to$ Jacobian

$\Rightarrow X_{ge} = Y_{ge}$

$\Rightarrow X_g = Y_g$ [e & g → be pts]

$= \boxed{X = Y}$ $\forall g \in G$.

$\Rightarrow i_L$ is 1-1

$\dim G = \dim \mathcal{G}$

② i_L is onto:

Let $X_e \in T_e$ then $\exists\, X \in \mathcal{G}$ s.t. $i_L(X) = X_e$.
Hence i_L is onto.

Theorem

③ i_L is linear trasfm / Homo

Let $\alpha, \beta \in \mathbb{R}$, $X, Y \in \mathcal{G}$

$i_L(\alpha X + \beta Y) = (\alpha X + \beta Y)_e = \alpha X_e + \beta Y_e = \alpha\, i_L(X) + \beta\, i_L(Y)$
$\Rightarrow i_L$ is lin. trasf.

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Lie transformation grp:

Let $G \to$ Lie grp.

$M \to$ ~~...~~ C^∞ n-manifold.

Then G is ~~a~~ a Lie ~~grp~~ transformation grp of M if there is a diff^ble map

$\psi : M \times G \to M$ defined as

$\psi(x,g) = xg \quad \forall x \in M, g \in G.$

s.t. $(xg)h = x(gh)$, $h \in G$.

& $xe = x$.

In this case, we say that G acts on M to the right or G acts differentiably to ~~M~~ M from Right.

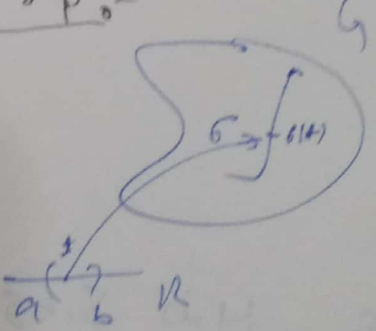
Similarily action of G to M from left can be defined by $\phi : G \times M \to M$. ~~st~~ given by

$\phi(g,x) = gx$.

s.t. $h(gx) = (hg)x$.

& $ex = x$.

One parameter Subgrp of Lie grp:-



Let $G \to$ Lie grp.

Let σ be a C^∞ curve in G.

i.e $\sigma : (a,b) \to G$, is a diff^ble mapping where $(a,b) \subset \mathbb{R}$

Consider $H = \{\sigma(t) \mid t \in \mathbb{R}\} \Rightarrow$ Set of pts on the curve σ in G.

obviously $H \subset G$.

claim: H is Subgrp of G.

For; Define '$\circ$' in H as

$\sigma(s) \circ \sigma(t) = \sigma(s+t)$ as $s, t \in \mathbb{R}$

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A-D:

So H is closed under opr 'o'.

Existence of Identity:

since $0 \in R$

$\sigma(0) \in H$

let $\sigma(t) \in H$

$\sigma(0) \circ \sigma(t) = \sigma(0+t) = \sigma(t)$

& $\sigma(t) \circ \sigma(0) = \sigma(t+0) = \sigma(t)$

$\Rightarrow$ $\sigma(0)$ is identity elt of H.

Existence of Inverse:

let $t \in R$

$\Rightarrow -t \in R$

i.e $\sigma(t) \in H$ & $\sigma(-t) \in H$

& $\sigma(t) \circ \sigma(-t) = \sigma(t-t) = \sigma(0)$

& $\sigma(-t) \circ \sigma(t) = \sigma(-t+t) = \sigma(0)$

Thus $\sigma(-t)$ is Inv elt of $\sigma(t)$ in H.

Associativity:

$$\{\sigma(s) \circ \sigma(t)\} \circ \sigma(r) = \{\sigma(s+t)\} \circ \sigma(r)$$
$$= \sigma\{(s+t)+r\} \quad \because r, s, t \in R \text{ Assoc}$$
$$= \sigma\{s+(t+r)\}$$
$$= \sigma(s) \circ \sigma(t+r)$$
$$= \sigma(s) \circ \{\sigma(t) \circ \sigma(r)\}$$

Therefore H is Subgrp of G wrt 'o'.

H is c/a One parameter Subgrp of the grp G.

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Some problems:

Ex ① In a die grp G, $\forall g \in G$, P.T. the map $I_g : G \to G$ defined as $I_g(h) = ghg^{-1}$ is an automorphism of G.

Pf: Claim: $I_g : G \to G$ is an Isomorphism

For:

① I_g is 1-1:

let $h_1, h_2 \in G$.

If $I_g(h_1) = I_g(h_2)$

$\Rightarrow gh_1g^{-1} = gh_2g^{-1}$

$\Rightarrow gh_1 = gh_2$

$\Rightarrow \boxed{h_1 = h_2}$

② I_g is Onto:

let $h \in G$ (CoDomain)

let $I_g(x) = h$, where $x \in G$ (Domain)

$gxg^{-1} = h$

$\boxed{x = g^{-1}hg.}$

i.e $I_g(g^{-1}hg) = h$.

So I_g is Onto.

③ I_g Preserves the Composition:

let $h_1, h_2 \in G$.

$I_g(h_1h_2) = g(h_1h_2)g^{-1}$

$= gh_1(g^{-1}g)h_2g^{-1}$

$= (gh_1g^{-1})(gh_2g^{-1})$

$= I_g(h_1) \cdot I_g(h_2)$

Proved:

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Ex ② PT.
① $L_g \circ L_h = L_{gh}$ ② $L_g \cdot R_h = R_h L_g$ ③ $R_g R_h = R_{hg}$

Pf: ① let $t \in G$

$(L_g \circ L_h)(t) = \cancel{L_g(} L_g(L_h(t))$
$= L_g(ht)$
$= ght$
$= L_{gh}(t) \quad , \forall t$

$\Rightarrow \boxed{L_g \circ L_h = L_{gh}}$

② $L_g R_h(t) = L_g(th) = gth$
$= R_h(gt)$
$= R_h \cdot L_g(t)$
$\forall t$

$\Rightarrow L_g R_h = R_h L_g$

③ $R_g R_h(t) = R_g(th) = (th)g = t(hg)$ (By Asso)
$= \cancel{R_g(t)}$ $\cancel{= t(}$
$= R_{hg}(t) \quad , \forall t.$

$\Rightarrow R_g R_h = R_{hg}.$

Ex ③: PT! ① $C_x \circ C_y = -C_y \circ C_x.$
② If $\alpha \in F^p(A)$, $\beta \in F^q(A)$, Then $(\alpha \wedge \beta) = (-1)^{pq}(\beta \wedge \alpha)$

Pf ① LHS! $\omega \in \cancel{F^p(A)}\ F^p(A)$
let $x_1, x_2, \ldots x_{p-1} \in T^{1,0}(A)$

$\cancel{C_x \circ C_y}$:
$\{(C_x \circ C_y)\omega\}(x_1, x_2, \ldots x_{p-1}) = C_x(C_y\omega)(x_1, x_2, \ldots x_{p-1})$
$= C_x\{\omega(y, x_1, x_2, \ldots x_{p-1})\}$
$= \omega(x, y, x_1, x_2, \ldots x_{p-1})$
$= -\omega(y, x, x_1, x_2, \ldots x_{p-1})$
$= -(C_y \circ C_x\omega)(x_1, x_2, \ldots x_{p-1})$

or $\boxed{C_x \circ C_y = -C_y \circ C_x}$

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Unit - IV

Principal Fibre Bundle:

A set $\{P, M, G, \pi\}$ is called principal fibre bundle if P & M are diff'ble Manifold, G is a Lie ~~grp~~ transformation on M from the right. π is a projection map, satisfying following conditions: —

1. G acts on P differentiably to the right, i.e $\exists$ a diff'ble map from $P \times G \to P$ s.t
$$(u, g) = ug \quad , \forall u \in P, \; g \in G. \quad ug \in P$$
and
$$(ug)h = u(gh) \quad , \forall h \in G.$$

2. M is a quotient manifold, i.e $M \cong P/G$ and π is a projection map $\pi: P \to P/G$ $(\pi: P \to M)$ & is diff'ble.

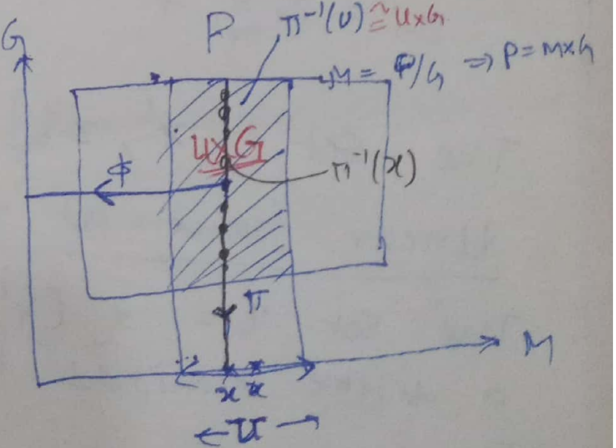
3. For each $x \in M$, & for every nbd U of x such that $\pi^{-1}(U)$ is isomorphic to $U \times G$ by the diffeomorphism
$$\psi: \pi^{-1}(U) \to U \times G$$
s.t $\psi(u) = (\pi(u), \phi(u))$, where $\phi: \pi^{-1}(U) \to G$ satisfying $\phi(ug) = \phi(u)g$ $\forall u \in \pi^{-1}(U)$ $\forall g \in G$.

We call P → Total Space or Bundle Space

M → Base Space

G → Structure grp.

π → Projection map.



$\forall x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P, called fibre over x.

or

If $u \in \pi^{-1}(x)$; Then $\pi^{-1}(x)$ is the set of point (u, g), $g \in G$ $= ug$ and is called fibre through u.

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Linear frame:

Let $M \to$ Diff^{ble} Manifold.

& Let $x \in M$.,

$U \to$ nbd of x.

& $(x^1, x^2, \dots x^n) \to$ LCS in V

The Canonical basis vectors in V are $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots \frac{\partial}{\partial x^n}\}$

Let $\{z_1, z_2, z_3, \dots z_n\}$ be another basis in V. Then

we can write -

$$z_1 = z_1^1 \frac{\partial}{\partial x^1} + z_1^2 \frac{\partial}{\partial x^2} + z_1^3 \frac{\partial}{\partial x^3} + \dots + z_1^n \frac{\partial}{\partial x^n}$$

$$z_2 = z_2^1 \frac{\partial}{\partial x^1} + z_2^2 \frac{\partial}{\partial x^2} + z_2^3 \frac{\partial}{\partial x^3} + \dots + z_2^n \frac{\partial}{\partial x^n}$$

$$\vdots$$

$$z_n = z_n^1 \frac{\partial}{\partial x^1} + \dots \dots + z_n^n \frac{\partial}{\partial x^n}$$

The set (x^i, z_b^a), $1 \le i \le n$, $1 \le a, b \le n$ is called

Linear frame at x.

The set $L = \{(x^i, z_b^a)\}$ of all linear frames is a diff^{ble} Manifold of dim. $n + n^2$

Doubt Claim: $\{L, M, GL(n,\mathbb{R}), \pi\} \to$ Principal fibre bundle
f.c.a
Linear frame bundle.

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P-3

(i) First of all, we'll show that $Gl(n,R)$ acts diff'ly on L to the right.

~~let $u \in L$, $g \in Gl(n,R)$~~ given by $u \equiv (x^i, z^a_b)$, $g = g^b_c$

Fn, Define a map, from $L \times Gl(n,R) \to L$

$(u,g) \to ug.$

st $\left((x^i, z^a_b), g^b_c\right) = \left(x^i, z^a_b g^b_c\right) \in L$

where $(x^i, z^a_b) \in L$ & $g^b_c \in Gl(n,R)$

<u>Observe:</u> $(ug)h = u(gh)$

i.e. $\left(\left[x^i, z^a_b\right] \cdot g^b_c\right) \cdot h^c_d = \left(x^i, z^a_b g^b_c\right) h^c_d$

$= \left(x^i, z^a_b \left(g^b_c h^c_d\right)\right)$

$= (x^i, z^a_b)\left(g^b_c h^c_d\right)$

$= u(gh)$

So $Gl(n,R)$ acts differentiably on L to the right.

(ii) We can take $M = L / Gl(n,R)$, & π as projection Map

$\pi : L \to M$ s.t

$u = (x^i, z^a_b) \in L$, $\pi(x^i, z^a_b) = x^i$, which is diff'al Map.

(iii) let $x \in M$, $U \to$ nbd of x in M.

Then $\pi^{-1}(U) \subset L = M \times Gl(n,R)$

let $\pi^{-1}(U) = \{(x^i, z^i_b) \mid x^i \in M, z^i_b \in Gl(n,R)\}$

Consider a map from;

$(x^i, z^i_a) \xrightarrow{Id} (x^i, z^i_a)$ which is a identity Map

from $\pi^{-1}(U) \xrightarrow{Id} U \times Gl(n,R)$. obviously Id. map is isomap

So $\pi^{-1}(U) \cong U \times Gl(n,R)$. So the set $\{ - \}$ is principal fibre bundle

P.4

Tangent Bundle

Let $M \to n$ dim diff'ble manifold at each pt $p \in M$,

Then, $\exists$ n dimensional tangent Sp T_pM, then

$$TM = \bigcup_{p \in M} T_pM,$$ Coll^n of all Tangent Spaces of M is called Tangent bundle of M

TM $\to$ Diff'ble Manifold of dim $\underline{\underline{2n}}$.

The Manifold M, over which TM is defined is called base space

Quest: Prove that, Tangent Bundle TM is always orientable.

Pf: Let M be a diff'ble Manifold and TM $\to$ Tangent Bundle

Let π is the Projection Map $\pi : TM \to M$.

Let $x \in M$ & U be nbd of pt x in M.

Let $\{x^1, x^2, \ldots x^n\} \to$ Local coord. Syst in U

we can also write this coordinate —

(x^h), $h = 1, 2, \ldots n$.

Then $\pi^{-1}(U)$ is the ~~local~~ coord. nbd ~~Syst~~ in TM, with Coord Syst $\{x^1, x^2, \ldots x^n, y^1, y^2, \ldots y^n\}$ or (x^h, y^h), $h = 1, 2, \ldots n$.

Let $U' \to$ another ~~loc~~ nbd of x in M with LCS $(x^{1'}, x^{2'}, x^{3'} \ldots x^{n'})$ or $x^{h'}$, $h' = 1', 2', \ldots n'$

Then $\pi^{-1}(U')$ is Coord ~~co~~ nbd in TM, with local Coord Syst $\{x^{1'}, x^{2'}, x^{3'}, \ldots x^{n'}, y^{1'}, y^{2'}, \ldots y^{n'}\}$, or $(x^{h'}, y^{h'})$ $h' = 1', 2', \ldots n'$.

Since $U \cap U' \neq \phi$

$\Rightarrow \pi^{-1}(U) \cap \pi^{-1}(U') \neq \phi$

Now, for the points which ~~lying~~ in the region — $\pi^{-1}(U) \cap \pi^{-1}(U')$

we have

$$\left.\begin{aligned} x^{h'} &= x^{h'}(x^h) \\ \& \; y^{h'} &= \frac{\partial x^{h'}}{\partial x^h} y^h \end{aligned}\right\} \quad \text{—— (A)}$$

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P-5 putting $y^i = x^{n+i}$

or $y^1 = x^{n+1}$

$y^2 = x^{n+2}, \; --- \; y^n = x^{2n}$

So local coord. syst in $\pi^{-1}(U)$ is (x^p), $p = 1, 2, --- n, n+1, -- 2n$

& local coord syst in $\pi^{-1}(U')$ is $(x^{p'})$, $p' = 1', 2', -- n', n'+1, --2n'$

Eq^n (A), May also be expressed as —

$$\left.\begin{aligned} x^{h'} &= x^{h'}(x^h) \\ \& \; y^{h'} &= \frac{\partial x^{h'}}{\partial x^i} y^i \end{aligned}\right\} \text{— (B)}$$

Now, The Jacobian Mat^x of transformation, for the pts lie in the region $\pi^{-1}(U) \cap \pi^{-1}(U')$ is given by.

$$\left(\frac{\partial x^{p'}}{\partial x^p}\right) = \frac{\partial (x^{h'}, y^{h'})}{\partial (x^h, y^h)} = \begin{bmatrix} \frac{\partial x^{h'}}{\partial x^h} & \frac{\partial x^{h'}}{\partial y^h} \\ \frac{\partial y^{h'}}{\partial x^h} & \frac{\partial y^{h'}}{\partial y^h} \end{bmatrix}.$$

Since $\frac{\partial x^{h'}}{\partial y^h} = \frac{\partial}{\partial y^h}\left(x^{h'}(x^h)\right)$

$= 0$

$$\frac{\partial y^{h'}}{\partial x^h} = \frac{\partial}{\partial x^h}\left(\frac{\partial x^{h'}}{\partial x^i} y^i\right) = \frac{\partial^2 x^{h'}}{\partial x^h \partial x^i} y^i$$

$$\frac{\partial y^{h'}}{\partial y^h} = \frac{\partial}{\partial y^h}\left(\frac{\partial x^{h'}}{\partial x^h} y^h\right) = \frac{\partial x^{h'}}{\partial x^h} + 0$$

$$= \frac{\partial x^{h'}}{\partial x^h}$$

Thus
$$\frac{\partial (x^{h'}, y^{h'})}{\partial (x^h, y^h)} = \begin{bmatrix} \frac{\partial x^{h'}}{\partial x^h} & 0 \\ \frac{\partial^2 x^{h'}}{\partial x^h \partial x^i} y^i, & \frac{\partial x^{h'}}{\partial x^h} \end{bmatrix} \text{ — (1)}$$

~~Obviously~~ ~~Jacobian~~ Det of Jacobian Mat^x is non zero.

Now, we can also consider, the reverse —

$x^h = x^h(x^{h'})$, $y^h = \frac{\partial x^h}{\partial x^{i'}} y^{i'}$

Then Jacobian Mat^x of transformation is given by

$$\frac{\partial (x^h, y^h)}{\partial (x^{h'}, y^{h'})} = \begin{bmatrix} \frac{\partial x^h}{\partial x^{h'}} & \frac{\partial x^h}{\partial y^{h'}} \\ \partial y^h/\partial x^{h'} & \frac{\partial y^h}{\partial y^{h'}} \end{bmatrix}$$

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P-t/

Since

$$\frac{\partial y^h}{\partial x^{h'}} = \frac{\partial}{\partial x^{h'}}\left(\frac{\partial x^h}{\partial x^{i'}} y^{i'}\right)$$

$$= \frac{\partial^2 x^h}{\partial x^{h'}\partial x^{i'}} y^{i'} + 0$$

$$\frac{\partial y^h}{\partial y^{h'}} = \frac{\partial}{\partial y^{h'}}\left(\frac{\partial x^h}{\partial x^{h'}} y^{h'}\right) = \frac{\partial x^h}{\partial x^{h'}}.1 + 0$$

$$= \frac{\partial x^h}{\partial x^{h'}}$$

Thus

$$\frac{\partial(x^h, y^h)}{\partial(x^{h'}, y^{h'})} = \begin{bmatrix} \frac{\partial x^h}{\partial x^{h'}} & 0 \\ \frac{\partial^2 x^h}{\partial x^{h'}\partial x^{i'}} y^{i'} & \frac{\partial x^h}{\partial x^{h'}} \end{bmatrix} \quad \text{—— (2)}$$

Obviously det of (2) is non zero

Therefore we can say that Tangent Bundle is always orientable because its det. is non zero.

Vertical lift:

Let M be a diffble Manifold of dim n.

& $f: M \to \mathbb{R}$ (Real valued f^n)

Let $\pi: TM \to M$ (Projection Map)

Define $f^v = f \circ \pi : TM \to \mathbb{R}$, called the vertical lift of f.

$$\begin{tikzcd} TM \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi = f^v"] & \\ M \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

Quest: For a C^∞ f^n f & g on M, P.T.

(i) $(f+g)^v = f^v + g^v$

(ii) $(f.g)^v = f^v g^v$

Pf: (i) $(f+g)^v = (f+g)\circ\pi = (f\circ\pi) + (g\circ\pi)$
$= f^v + g^v$

(2) $(f.g)^v = (f.g)\circ\pi = (f\circ\pi)(g\circ\pi)$
$= f^v g^v$

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P-7: Vertical lift of vector field:

Let $X \to$ vector field on M

Vertical lift of X is denoted by X^v & is defined as

$$X^v(T\omega) = (\omega(X))^v$$

where $T\omega$ is a C^∞ f^n in TM & ω is 1-from on M.

& $T: TM \to T_pM$ & $\omega: T_pM \to R$

i.e $T\omega: TM \to R$

Remark: $\boxed{X^v f^v = 0}$

Quest: PT

1. $(X+Y)^v = X^v + Y^v$
2. $[X^v, Y^v] = 0$

Pf: 1.
$$\begin{aligned}(X+Y)^v(T\omega) &= (\omega(X+Y))^v \\ &= (\omega(X)+\omega(Y))^v \\ &= (\omega(X))^v + (\omega(Y))^v = X^v(T\omega) + Y^v(T\omega) \\ &= (X^v+Y^v)(T\omega), \quad \forall T\omega\end{aligned}$$

$\Rightarrow (X+Y)^v = X^v + Y^v$

2.
$$\begin{aligned}[X^v, Y^v](T\omega) &= X^vY^v(T\omega) - Y^vX^v(T\omega) \\ &= X^v(\omega(Y))^v - Y^v \cancel{X(\omega)}\,(\omega(X))^v \\ &= 0 - 0 \\ &= 0 \qquad (\because X^v f^v = 0)\end{aligned}$$

Vertical lift of covariant Tensor of rank '2'

Let G be a covariant tensor of rank 2, with components G_{ij}

Then ~~G_{ij}~~ G can be be locally expressed as

$$G = G_{ij}\, dx^i \otimes dx^j$$

Taking vertical lift on both sides—

$$\begin{aligned}G^v &= (G_{ij})^v (dx^i)^v \otimes (dx^j)^v \\ &= (G_{ij})^v\, dy^i \otimes dy^j\end{aligned}$$

$\left(\because (dx^i)^v = dy^i \ \& \ \left(\frac{\partial}{\partial x^i}\right)^v = \frac{\partial}{\partial y^i}\right)$

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P-8'

Vertical lift of Contravariant Tensor of rank 2

let $H \rightarrow$ Contravariant Tensor of rank 2 with comp. H^{ij}

Then
$$H = H^{ij}\left(\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}\right)$$

Taking vertical lift :
$$H^V = (H^{ij})^V \left(\frac{\partial}{\partial x^i}\right)^V \otimes \left(\frac{\partial}{\partial x^j}\right)^V$$

$$\boxed{H^V = (H^{ij})^V \frac{\partial}{\partial y^i} \otimes \frac{\partial}{\partial y^j}}$$

Vertical lift of Tensor of type (1,1)

$$F = F_i^j \, dx^i \otimes \frac{\partial}{\partial x^j}$$

$$F^V = (F_i^j)^V (dx^i)^V \otimes \left(\frac{\partial}{\partial x^j}\right)^V$$

$$\boxed{F^V = (F_i^j)^V \, dy^i \otimes \frac{\partial}{\partial y^j}}$$

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