Basic principle of operation of FEL

The equation for energy exchange between electron and electromagnetic wave is

$$\dot{\gamma} = \frac{e\,\vec{\beta}.\vec{E}}{mc} \qquad \text{—(1)}$$

β = velocity of electron, $\gamma = 1/\sqrt{1-\beta^2}$ relativistic factor.

$\vec{E} = \vec{E}\sin(kz-\omega t)$

- Consider an electron and a transverse emw propagating colinearily, in the z direction in vacuum. Electric field of emw is orthogonal to the velocity of electron so that no work is done on electron.

- To generate an axial component of wave's electric field, consider propagation in a hollow metallic waveguide. For example, a TM wave in a rectangular guide will have ~~an~~ E_z and E_x both.

The energy equation for electron $\qquad z = c\beta_z t + z_0$

$$\dot{\gamma} = \frac{e\beta_z E_z}{mc}\sin\left[ck(\beta_z-\beta_\phi)t+\phi_0\right] \qquad \text{—(2)}$$

here $\beta_\phi = \omega/ck$. is the ~~wave~~ phase velocity of the wave.

- $\beta_z < \beta_\phi$ always and hence electron will slip in phase relative to the wave and the time averaged energy exchange between electron and emw will be zero.

- If the waveguide is loaded with a dielectric medium or a periodic perturbation, so called slow-wave structures, one can have axial electric field and a phase velocity less than c, resulting in a net energy exchange. ~~This is the principle of operation of a microwave~~ tube.

This mechanism ceases to provide good coupling between electron and wave ~~for~~ radiation wavelength shorter than 1 mm.

for infra red wavelengths and beyond, a new mechanism for coupling the electron to the wave is needed. This is where the free electron laser enters the picture.

- ~~slow wave devices rely on an axial electric field.~~

FEL makes use of a periodic magnetic field to impart a transverse motion to the electron and thereby couple it to the transverse field of a free emw.

An idealized planar undulator has a magnetic field of the form
$$\vec{B} = B_U \cos\left(\frac{2\pi z}{\lambda_U}\right) \vec{y} \qquad \text{— (3)}$$

λ_U is the period of alternating polarity field.

It is also possible to have a helical undulator with a field
$$\vec{B} = B_U \left[\cos\left(\frac{2\pi z}{\lambda_U}\right)\vec{x} + \sin\left(\frac{2\pi z}{\lambda_U}\right)\vec{y}\right] \qquad \text{— (4)}$$

Transverse velocity of the electron is of the form
$$\beta = -\frac{K}{\gamma} \sin\left(\underbrace{k_U c \beta_z t}_{k_U z}\right)\vec{x} \qquad \text{— (5)}$$
in a planar undulator.

and $$\beta = \frac{K}{\gamma}\left[\cos(k_U c\beta_z t)\vec{x} + \sin(k_U c\beta_z t)\vec{y}\right] \qquad \text{— (6)}$$
in a helical undulator.

here $$K = \frac{eB_U}{mc^2 k_U} = 0.934\, B_U\, [T]\, \lambda_U (cm) \qquad \text{— (7)}$$

is called undulator parameter and k_U is undulator wavenumber $k_U = 2\pi/\lambda_U$.

For an electron in a helical undulator interacting with a circularly polarized wave $\vec{E} = E_0 [\sin(kz-\omega t)\vec{x} + \cos(kz-\omega t)\vec{y}]$, the energy equation becomes
$$\dot{\gamma} = \frac{eE_0 K}{mc\gamma} \sin\left[(k+k_U)(c\beta_z t + z_0) - \omega t\right] \qquad \text{— (8)}$$

here z_0 is the initial electron position.

A similar eqn. holds for a planar undulator and a linearly taking due consideration of ponderomotive wave.

The electron can be thought of as interacting with a wave with frequency ω but with wave number $k+k_U$, the so called ponderomotive wave or beat wave.

Again, for a net energy exchange the phase must be stationary, which implies the following synchronism condition:

$$\beta_z = \frac{\omega/c}{k+k_U} \qquad \text{--- (9)}$$

i.e. the electron velocity equals the phase velocity of the ponderomotive wave.

while the emw has a phase velocity c, pmw has a phase velocity less than c, providing for synchronism between the electron and the wave.

Electron slips in phase w.r.t. emw exactly one light wavelength for each undulator period, but maintains a constant phase with r.r.t. pmw. The absolute value of the phase determines whether the electron is decelerated or accelerated.

If the synchronism condition is satisfied, the electron radiates emw with a frequency given by (9)

$$\omega = \frac{k_U c \beta_z}{1-\beta_z} \qquad \text{--- (10)}$$

for relativistic particles

$$\omega = \frac{2ck_U\gamma^2}{1+K^2} \qquad \text{--- (11)}$$

In practical units, radiated wavelength can be written as

$$\lambda(\text{Å}) = \frac{13.05\,\lambda_U(\text{cm})}{E^2(\text{GeV})}(1+K^2) \qquad \text{--- (12)}$$

ex. a linac driven expt. $E = 50\,\text{MeV}$, $K=1$, $\lambda_U = 10\,\text{cm}$,
$\Rightarrow \lambda = 10\,\mu\text{m}$ radiation.

When (9) is satisfied, the energy equation can be written as

$$\dot{\gamma} = \frac{eE_0 k}{mc\gamma} \sin\left[(k+k_U) z_0\right] \qquad \text{—(13)}$$

From this equation it follows that an electron can lose or gain energy according to the value of its phase $(k+k_U)z_0 = 2\pi z_0/\lambda_r$, with respect to the field. If there are a number of electrons in radiation wavelength λ_r with a uniform distribution in z_0, some of them will lose energy and some will gain energy. This change in energy produces change in longitudinal velocity of the electrons in the undulator, so that after some distance the electrons tend to regroup, or bunch, at the same phase w.r.t. the field.

For a bunched electron beam the radiation emitted by each electron is in phase and the total intensity, at the frequency given by (11) can be enhanced by a factor equal to the number of electrons.

Electron Trajectory in FEL

Consider a monoenergetic beam of electrons entering the FEL cavity along the z-axis with relativistic velocity $c\beta_0\hat{z}$ and energy $\gamma_0 mc^2$ ($\beta_0 \sim 1$). The beam propagates through a circularly polarized wiggler magnetic field of strength B, wavelength $\lambda_0 = 2\pi/k_0$ and wiggler frequency $\omega_0 = ck_0$, k_0 is wiggler wave number.

The helical magnetic field is represented by

$$\vec{B}_m = B(\cos k_0 z, -\sin k_0 z, 0) \qquad \text{—— (1)}$$

Electron Trajectory

The electron motion in the magnetic field is governed by Lorentz force equation

$$\cancel{dt}\ \frac{d}{dt}(\gamma\vec{\beta}) = \frac{e}{mc}(\vec{\beta}\times\vec{B}_m) \qquad \text{—— (2)}$$

$P = \gamma mc^2$

$\frac{dP}{dt} = \frac{e}{c}(\vec{v}\times\vec{B})$

$$\frac{d\gamma}{dt} = 0 \qquad \text{—— (3)}$$

Application of magnetic field does not change electron energy γmc^2. Thus $\gamma = \gamma_0 =$ constant —(3a)

Eqn.(2) may be written as

$$\frac{d\beta}{dt} = \frac{e}{mc\gamma_0}(\vec{\beta}\times\vec{B}_m) \qquad \text{—— (4)}$$

Put (1) in (4)

$$\vec{\beta}\times\vec{B}_m = \cancel{B}\begin{vmatrix} x & y & z \\ \beta_x & \beta_y & \beta_z \\ B_x & B_y & B_z \end{vmatrix}$$

$B_x = B\cos k_0 z$, $B_y = -B\sin k_0 z$, $B_z = 0$

$$\frac{d\beta_x}{dt} = \frac{e}{mc\gamma_0}\beta_z B\sin k_0 z \qquad \text{—— (5)}$$

$$\frac{d\beta_y}{dt} = \frac{e}{mc\gamma_0}\beta_z B\cos k_0 z \qquad \text{—— (6)}$$

$$\frac{d\beta_z}{dt} = -\frac{eB}{\gamma_0 mc}\left[\beta_x \overset{\sin k_0 z}{\cancel{\cos k_0 z}} + \beta_y \cos k_0 z\right] \qquad \text{—— (7)}$$

Let the initial conditions be (at $t=0$)

$$\left.\begin{aligned} \beta_x &= \beta_{x_0} \\ \beta_y &= \beta_{y_0} \\ z &= z_0 \\ \gamma &= \gamma_0 (1-\beta_0^2)^{1/2} \end{aligned}\right\} \quad (8)$$

Integrating equations (5) to (7) and using (8)

$$\beta_x = -\frac{K}{\gamma_0}\left[\cos k_0 z - \cos k_0 z_0\right] + \beta_{x_0} \quad \text{—— (9)}$$

$$\beta_y = \frac{K}{\gamma_0}\left[\sin k_0 z - \sin k_0 z_0\right] + \beta_{y_0} \quad \text{—— (10)}$$

here $K = \dfrac{eB}{mc^2 k_0} = \dfrac{eB}{mc\omega_0}$ called Wiggler parameter —— (11)

Put $\beta_{x_0} + \dfrac{K}{\gamma_0}\cos k_0 z_0 = D$ —— (12)

($\because$ these are constants)

set $\beta_{y_0} - \dfrac{K}{\gamma_0}\sin k_0 z_0 = 0$ —— (13)

Eqn (9) and (10) may be rewritten as

$$\beta_x = -\frac{K}{\gamma_0}\cos k_0 z + D \quad \text{—— (14)}$$

$$\beta_y = \frac{K}{\gamma_0}\sin k_0 z \quad \text{—— (15)}$$

$D=0$ corresponds to perfect beam injection yielding a perfect helical trajectory of constant radius K/γ_0 around z axis. Hence, D is a measure of deviation from perfect beam injection.

Apart from eqn (7) expression for β_z can also be obtained from

$$\frac{1}{\gamma_0^2} = 1 - \left(\beta_x^2 + \beta_y^2 + \beta_z^2\right) \quad \text{—— (16)}$$

Put eqn. (14) and (15) in eqn (16)

$$\beta_z = \beta_0' + \frac{KD}{\gamma_0}\cos k_0 z \qquad \text{——(17)}$$

$$\beta_0' = \left(1 - \frac{1}{\gamma_{11}}\right)^{1/2} \qquad \text{——(18)}$$

$$\gamma_{11}^2 = \gamma_0^2 / (1 + K^2 + \gamma_0^2 D^2) \qquad \text{——(19)}$$

Multiply eqn.(17) by c and integrate

$$z = z_0' + c\beta_0' t + \underbrace{\frac{KD}{\gamma_0 \beta_0' k_0}\sin k_0 z}_{\text{axial oscillation}} \quad \text{—(20)}$$

z_0' = constant of integration

for perfect beam injection $D = 0$

$$\beta_x = -\frac{K}{\gamma_0}\cos k_0 z$$

$$\beta_y = \frac{K}{\gamma_0}\sin k_0 z$$

$$\beta_z = \beta_0'$$

Spontaneous Emission

When electrons move under the influence of a periodic magnetic structure along the trajectory described above they experience acceleration and hence spontaneously emit electromagnetic radiation. The characteristics of this radiation are calculated using Fourier transform of the vector potential representing the spontaneous radiation field.

$$A(\omega_r) = \left(\frac{e^2\omega_r^2}{8\pi^2 c}\right)^{1/2} \int_{-\pi N/\omega_0\beta_0'}^{\pi N/\omega_0\beta_0'} \hat{z} \times (\hat{z} \times \vec{\beta}_\perp)\, e^{i\omega_r\left(t - \hat{z}\cdot\frac{\vec{r}}{c}\right)}\, dt \quad \text{—(21)}$$

here $\vec{\beta}_\perp = \hat{x}\beta_x + \hat{y}\beta_y$

ω_r = frequency of electromagnetic radiation
$\hat{z}$ = direction of observation.

The spontaneous energy radiated in the forward direction per unit solid angle ($d\Omega$) per unit frequency interval ($d\omega_r$) is given by

$$\frac{d^2 I}{d\Omega\, d\omega_r} \propto |A(\omega)|^2 \qquad \text{cavity length } L = N\lambda_0$$

$$\frac{d^2 I}{d\Omega\, d\omega_r} = 8\,\frac{(e\gamma_0 N)^2}{c} \sum_f \left\{ \frac{c^2 f^2}{L^2} \cdot \frac{K_{fs}(x)}{\{1+K^2+\gamma_0^2 D^2\}} \right\} \cdot \frac{\sin^2\{\omega_r(1-\beta_0') - f\omega_0\beta_0'\}\, \frac{\pi N}{\omega_0\beta_0'}}{\{\omega_r(1-\beta_0') - f\omega_0\beta_0'\}^2}$$

here $K_{fs}(x) = K\left[J_{f+1}^2(fx) + J_{f-1}^2(fx) - \frac{2(1+K^2)}{K^2} J_f^2(fx) \right]^{1/2}$

$K_{fs} \to$ coupling factor of electron motion and spontaneous radiation.

Now frequency of emitted radiation is obtained by resonance condition i.e.

$$\omega_r(1-\beta_0') - f(\omega_0\beta_0') \simeq 0.$$

$$\Rightarrow \omega_r = \frac{f\omega_0\beta_0'}{(1-\beta_0')} = f\omega_0\beta_0' \, \frac{2\gamma_0^2}{1+K^2+\gamma_0^2 D^2}$$

$f = 1, 2, 3, \ldots$ is the harmonic number.

These are the frequencies for which the emitted radiation will be maximum. Here we get more than one frequency

$$D = \beta x_0 + \frac{K}{\gamma_0}\cos k_v z$$

Interaction of electron trajectories with radiation field

Electron trajectories undergo small perturbations in the presence of electromagnetic radiation. This radiation may be from an external source or may be the spontaneous emission from the electrons themselves. As a result of interaction between the transverse velocity components of the electron's motion and the electromagnetic field there results a net gain or loss in the amplitude of the radiation field, depending upon the initial conditions. The analysis is simpler in the small gain per pass limit which means that the electric field can be taken to be practically constant over a single pass of the electron through the FEL cavity.

The shape of the electron trajectory determines the polarization of the cavity radiation.

For the helical wiggler magnet represented by

$$\vec{B}_m = B\,(\cos k_0 z,\ -\sin k_0 z,\ 0)$$

the electromagnetic radiation with matching polarization is given by

$$\left.\begin{aligned}\vec{E}_r &= E\,(\cos\xi,\ \sin\xi,\ 0)\\ \vec{B}_r &= E\,(-\sin\xi,\ \cos\xi,\ 0)\end{aligned}\right]\text{should satisfy Maxwell equations.}$$

here $\xi = k_r z - \omega_r t + \phi$; $\omega_r = c k_r$, $\lambda_r = 2\pi / k_r$

λ_r is the wavelength of radiation of strength E and phase ϕ. Both the fields match with each other in the sence that their rotation are in the same direction.

The motion of electrons in the electric and magnetic fields is given by

$$\frac{d\vec{p}}{dt} = \frac{d(\gamma m \vec{v})}{dt} = e\left[\vec{E} + \frac{1}{c}(\vec{v}\times\vec{B})\right]$$

$\omega_p^2 = 4\pi\rho$ measure of electron density.

The laser gain is given by

$$G(t) = \left[\frac{\omega_p^2}{\gamma_0} \beta_0^2 \frac{K^2}{\gamma_0^2} \frac{\omega_r}{2\gamma_0^2} \left(1 + \beta_0^2 + K^2\right) \cdot \frac{1}{(\Delta\omega)^3} \left[2 - 2\cos\Delta\omega t - \Delta\omega t \sin\Delta\omega t \right] \right]$$

here $K = \frac{U_B}{\omega_0 \beta_0} = \frac{eB}{mc\,\omega_0 \beta_0}$ $\qquad U_B = \frac{eB}{mc}$

For max^m $G(t)$

$\frac{1}{(\Delta\omega)^3}\left[2 - 2\cos\Delta\omega t - \Delta\omega t \sin\Delta\omega t\right]$ should be max^m

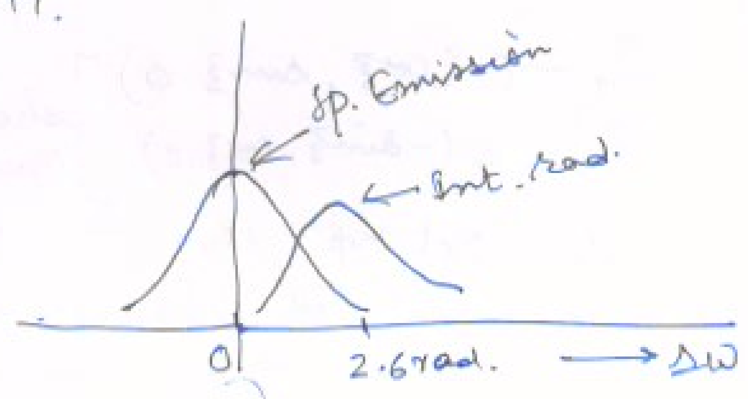
which gives $\Delta\omega = 2.6$ rad.

For obtaining detectable gain, $\Delta\omega$ must be small, i.e., the gain must be evaluated near resonance i.e. off resonance. $\Rightarrow$ 2.6 rad.

The resonance condition is obtained by putting $\Delta\omega \simeq 0$. $\qquad \Delta\omega = \omega_0\beta_0 - \omega_r(1-\beta_0) = 2.6$ rad.

$\Rightarrow \quad \omega_r = 2\gamma_0^2 \omega_0 \beta_0$ i.e, the Doppler upshifted wiggler frequency.

$\left.\begin{matrix}\beta_0 \\ \beta_0'\end{matrix}\right\} \simeq 0.99.$



FEL $\begin{cases} \text{Oscillatory mode} \\ \text{amplification mode.} \end{cases}$