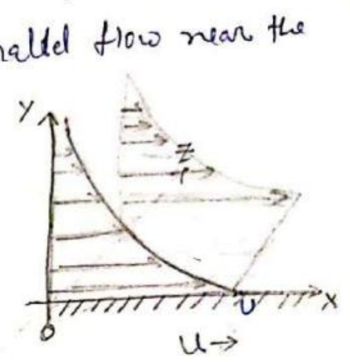
UNSTEADY MOTION of fluid Over ~~OF A~~ FLAT PLATE

The Simplest flow of this class is that which results when the motion of the Plate is started impulsively from rest. Consider an infinite Plate in an infinite fluid initially at rest and then suddenly given a Constant Velocity U in the x-direction. let the Plate is in X-Z Plane. This generates two dimensional (X-Z) Parallel flow near the Plate.



By Eqn of Continuity

$$\nabla \cdot \vec{q} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}^{\,0} + \cancel{\frac{\partial w}{\partial z}}^{\,0} = 0 \qquad (\because v = w = 0)$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 0 \qquad \text{—①}$$

$$\Rightarrow u = u(y, t)$$

Since the Plate is situated in an infinite fluid where the pressure is Constant every where therefore pressure gradients ~~ed~~ are Zero, So by N-stoke's Eqn ~~we~~ without body forces redues to

$$\rho \frac{D\vec{q}}{Dt} = \mu \nabla^2 \vec{q}$$

$$\rho \frac{Du}{Dt} = \mu \nabla^2 u \qquad (\because v = w = 0)$$

$$\rho \left(\frac{\partial u}{\partial t} + \underbrace{u \frac{\partial u}{\partial x}}_{\substack{0 \\ \text{by (1)}}} + \underbrace{v \frac{\partial u}{\partial y}}_{0} + \underbrace{w \frac{\partial u}{\partial z}}_{0} \right) = \mu \nabla^2 u$$

N-stokes Eqn for Incomp. fluid ~~with~~ with Constant Viscosity is

$$\rho \frac{D\vec{q}}{Dt} = \underbrace{\rho \cancel{F}}_{0} - \underbrace{\cancel{\nabla p}}_{0} + \mu \nabla^2 \vec{q}$$

$$\rho \frac{\partial u}{\partial t} = \mu \nabla^2 u$$

$$\rho \frac{\partial u}{\partial t} = \mu \left(\underset{0}{\cancel{\frac{\partial^2 u}{\partial x^2}}} + \frac{\partial^2 u}{\partial y^2} + \underset{0}{\cancel{\frac{\partial^2 u}{\partial z^2}}} \right)$$

$$\frac{\partial u}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \text{—} \quad ②$$

So we must solve this Eqn using boundary conditions

$u = 0$ when $t \leq 0$ $\forall y$ —(i)

$u = U$ at $y = 0$ } when $t > 0$ (ii)
$u = 0$ at $y = \infty$ }

the Partial diff. eqn (2) can be transformed into ordinary diff. Eqn by using transformation (Similarity Soln)

$$u = U \cdot f(\eta)$$

where, $\eta = \frac{y}{2\sqrt{\nu t}}$

now $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = U \cdot f'(\eta) \cdot \left\{ \frac{-y}{4\sqrt{\nu}} \cdot t^{-\frac{3}{2}} \right\}$

$$= -\frac{U}{2t} \cdot \frac{y}{2\sqrt{\nu t}} \cdot f'(\eta) = -\frac{U}{2t} \cdot \eta \cdot f'(\eta)$$

$$\frac{\partial u}{\partial y} = U f'(\eta) \frac{\partial \eta}{\partial y} = U. f'(\eta). \frac{1}{2\sqrt{\nu t}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U}{2\sqrt{\nu t}} f''(\eta). \frac{\partial \eta}{\partial y} = \frac{U}{2\sqrt{\nu t}} . f''(\eta). \frac{1}{2\sqrt{\nu t}} = \frac{U}{4\sqrt{\nu t}} f''(\eta)$$

Substituting $\frac{\partial u}{\partial t}$ & $\frac{\partial^2 u}{\partial y^2}$ in ② . we get

$$-\frac{\cancel{U}}{2\cancel{t}} \eta . f'(\eta) = \frac{\cancel{\nu}\cancel{U}}{4\cancel{\nu t}} f''(\eta)$$

$$\frac{f''(\eta)}{f'(\eta)} = -2\eta$$

$$\log f'(\eta) = -\frac{\cancel{2}\eta^2}{\cancel{2}} + \log c_1$$

$$\log \frac{f'(\eta)}{c_1} = -\eta^2$$

$$f'(\eta) = c_1 e^{-\eta^2}$$

$$\frac{df}{d\eta} = c_1 . e^{-\eta^2}$$

$$f = c_1 . \int_0^{\eta} e^{-\eta^2} d\eta + c_2 \qquad \text{——} ③$$

Boundary Conditions transformed to under the transformation of Variable $f(\eta) = \frac{u}{U}$, $\eta = \frac{y}{2\sqrt{\nu t}}$

by B.C (ii) when $y=0$ i.e $\underline{\eta = 0}$ $u = U$ i.e $\underline{f = 1}$

when $t=0$ ie $\underline{\eta = \infty}$ $\underline{f = \frac{u}{U} = 0}$

therefore by Eqn ③ we have

$$C_2 = 1$$

&

$$0 = C_1 \int_0^{\infty} e^{-\eta^2} d\eta + 1$$

$$\Rightarrow C_1 \int_0^{\infty} e^{-\eta^2} d\eta = -1$$

$$\Rightarrow C_1 = \frac{-1}{\int_0^{\infty} e^{-\eta^2} d\eta} = -\frac{2}{\sqrt{\pi}}$$

So,

$$f(\eta) = \frac{-2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta + 1$$

$$\frac{u}{U} = (1 - \text{erf}\,\eta)$$

$$\Rightarrow u = U\,(1 - \text{erf}\,\eta)$$

Shearing stress

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y}\right)$$

$$= \mu \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$= \mu U \cdot f'(\eta) \cdot \frac{1}{2\sqrt{\nu t}}$$

η	$\text{erf}\,\eta$
0	0
∞	1

$$\eta = \frac{y}{2\sqrt{\nu t}}$$

for u Vs t for any y.

$u = 0$ $t = 0$

$u = U$ $t = \infty$

So, as time t increases velocity of any ~~[illegible]~~ layer at any distance y increases up to limit U.

Shear stress at the Plate

$$(\tau_{yx})_{y=0} = \mu \frac{U \cdot f'(0)}{2\sqrt{\nu t}} = \frac{\mu U C_1}{2\sqrt{\nu t}} = -\frac{\mu U}{\sqrt{\nu \pi t}}$$