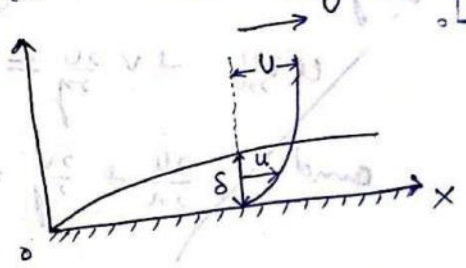
Flow parallel to a Semi-Infinite flat plate within the B. Layer. (The Blasius Sol^n)

Consider the steady flow of an incompressible viscous fluid past a thin flat plate which is placed in the direction of a uniform stream of velocity $U\hat{i}$. The plate is of ~~infinite length~~ finite length and much larger breadth. Hence problem is one of 2-dim motion which can be analyzed by using the ~~pran~~ Prandtl boundary layer Equations. Let the origin of the co-ordinates be at the leading edge of the plate, the x-axis be the direction of the uniform stream and the y axis normal to the plate.



The Prandtl boundary layer Equations in the case under Consideration are

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \cancel{\nu\frac{\partial^2 u}{\partial y^2}} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \right. \qquad (A)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$\frac{\partial p}{\partial y} = 0$$

... velocity in the

outside the B.L. we can approximate the flow as non viscous fluid flow. thus we have by Euler's eqn of motion

$$\frac{DU}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{— (1)}$$

for steady flow

$$U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{— (2)}$$

using this result in Prandtl's B.L. Eqns becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

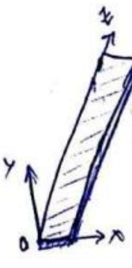
and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

If U is constant (for uniform stream)

Above Eqn (2) reduces to

$$\frac{\partial p}{\partial x} = 0$$



$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

p = Const. outside the B.L. and thus approximately uniform throughout the B.L. Hence for steady flow over a thin flat plate B.L. Eqn reduces to

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}}$$

~~... are zero - then~~

So. Eq^n (1) reduces to

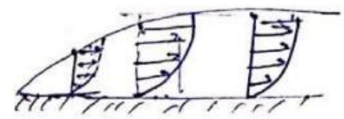
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{--- (3)}$$

with the B.C.

$u = v = 0 \quad ; y = 0 \quad$ — (i)

$u = U \quad ; y = \infty \quad$ — (ii)



we assume velocity profile as

$$u = U \cdot F(x, y, \nu, U) = U \cdot F(\eta) \quad \text{--- (4)}$$

(similarity sol^n.)
η - similarity variable

where η is dimensionless parameter. Since the B.L. ~~profile~~ will be fixed in 'shape' so obvious velocity profile will vary only in scale and ~~ch~~ choice of parameter $\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\frac{\nu x}{U}}} = y x^{-\frac{1}{2}}\sqrt{\frac{U}{\nu}}$ — (5)

$\left(\because o(\delta) = o\left(\frac{\nu x}{U}\right)\right)$

If ψ is stream function then we can take

$$u = \frac{\partial \psi}{\partial y} \quad \& \quad v = -\frac{\partial \psi}{\partial x}$$

$\because o\left(\frac{L^2}{\delta^2 R}\right) = o(1)$

$o(\delta^2) = o\left(\frac{L^2}{R}\right) = o\left(\frac{x^2}{\frac{Ux}{\nu}}\right)$

$$\psi = \underset{\text{treating } x \text{ const.}}{\int u\, dy} = \cancel{\int U F(\eta) \cdot \frac{\partial y}{\partial \eta}\, d\eta} = U\int F(\eta) \cdot \sqrt{\frac{x\nu}{U}}\, d\eta$$

(by eq^n 5)

$$\psi = \sqrt{U\nu x}\int F(\eta)\, d\eta = \sqrt{U\nu x}\,.\, f(\eta) \quad \text{--- (6)}$$

where, $f(\eta) = \int F(\eta)\, d\eta$

$$u = U F(\eta) = U.\, f'(\eta)$$

and

$$V = -\frac{\partial \psi}{\partial x} = -\sqrt{U\nu x}\; f'(\eta).\, \frac{\partial \eta}{\partial x} - \sqrt{U\nu}\, f(\eta).\left(\frac{1}{2}\frac{1}{\sqrt{x}}\right)$$

$$= -\sqrt{U\nu x}\; f'(\eta).\, y\sqrt{\frac{U}{\nu}}\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) - \frac{1}{2}\sqrt{\frac{U\nu}{x}}\, f(\eta)$$

$$= +\sqrt{U\nu x}\left(\frac{\eta}{2x}\right) f'(\eta) - \frac{1}{2}\sqrt{\frac{\nu U}{x}}\; f(\eta)$$

$$= \frac{1}{2}\sqrt{\frac{\nu U}{x}}\; \left[\eta f'(\eta) - f(\eta)\right]$$

we can evaluate

$$\frac{\partial u}{\partial x} = \frac{\partial^2 \psi}{\partial x \partial y} = -\frac{U}{2}\frac{\eta}{x} f''(\eta)$$

$$\frac{\partial u}{\partial y} = U\sqrt{\frac{U}{\nu x}}\; f''(\eta)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U^2}{\nu x}\; f'''(\eta)$$

Substituting u, v, u_x, u_y & u_{yy} in Eq^n (2) we get

$$-\frac{U^2}{2}\frac{\eta}{x} f' f'' + \frac{U^2}{2x}(\eta f' - f) f'' = \frac{U^2}{x} f'''$$

i.e $\quad 2f''' + f.f'' = 0 \quad$ — (7)

The B.C. transform ~~into~~ in terms of f & η are

(i) $\quad \eta = 0 \; : \quad f = 0 \;,\; \frac{df}{d\eta} = 0$

(ii) $\quad \eta \to \infty \; : \quad \frac{df}{d\eta} \to 1$

Eq^n (7) is usually referred as the Blasius ~~eq~~ eq^n.

Eqn is 3rd order non linear diff. eqn and no closed form soln has been found. Blasius in 1908 obtain the soln in the form of Power series about $\eta=0$.

$$f = A_0 + A_1\eta + \frac{A_2}{2!}\eta^2 + \frac{A_3}{3!}\eta^3 + \cdots$$

$$f' = A_1 + A_2\eta + \frac{A_3}{2!}\eta^2 + \frac{A_4}{3!}\eta^3 + \cdots$$

$$f'' = A_2 + A_3\eta + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 + \cdots$$

$$f''' = A_3 + A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \cdots$$

using B.C. (i) at $\eta=0$ to eqn for f & f'. we get

$$A_0 = 0 \quad ; \quad A_1 = 0$$

Substituting the ~~result~~ value thus obtained for f, f'', f''' into ⑦ we find

$$2A_3 + (2A_4)\eta + (A_2^2 + 2A_5)\frac{\eta^2}{2!} + (4A_2A_3 + 2A_6)\frac{\eta^3}{3!} + \cdots = 0$$

Equating Cofficient both side

we get $A_3 = A_4 = A_6 = A_7 = 0$

$$A_5 = -\frac{A_2^2}{2}$$

$$A_8 = \frac{11}{4}A_2^3 \quad \cdots$$

So,

$$f = \frac{A_2}{2!}\eta^2 - \frac{1}{2}\frac{A_2^2}{5!}\eta^5 + \frac{1}{4}\frac{11}{8!}A_2^3\eta^8 - \frac{1}{8}\cdot\frac{375}{11!}A_2^4\eta^{11} + \cdots$$

this eqn satisfies B.C (i) at $\eta=0$. and Constant A_2

will be determined from the B.C at $\eta = \infty$

Expression for f can be written as

$$f = A_2^{\frac{1}{3}}\left[\frac{(A_2^{\frac{1}{3}}\eta)^2}{2!} - \frac{1}{2}\frac{(A_2^{\frac{1}{3}}\eta)^5}{5!} + \frac{1}{4}\cdot\frac{11}{8!}(A_2^{\frac{1}{3}}\cdot\eta)^8 - \cdots\right]$$

$$f(\eta) = A_2^{\frac{1}{3}}\cdot G(A_2^{\frac{1}{3}}\eta)$$

now using B.C. at $\eta = \infty$

$$\lim_{\eta\to\infty} f'(\eta) = A_2^{\frac{2}{3}}\cdot \lim_{\eta\to\infty} G'(A_2^{\frac{1}{3}}\eta)$$

$$1 = A_2^{\frac{2}{3}}\cdot \lim_{\eta\to\infty} G'(\eta)$$

$$A_2 = \left[\lim_{\eta\to\infty}\frac{1}{G'(\eta)}\right]^{\frac{3}{2}}$$

we can determine A_2 numerically ~~0.33206~~ which is

$$A_2 = 0.33206.$$

Also we find $f''(0) = A_2 = 0.33206$

Shearing stress: at the Plate:

$$\tau_0 = \left(\mu\frac{\partial u}{\partial y}\right)_{y=0} = \left(\mu\frac{\partial u}{\partial y}\right)_{\eta=0}$$

$$= \mu\left(U f''(\eta)\sqrt{\frac{U}{\nu x}}\right)_{\eta=0}$$

$$= \mu U f''(0)\sqrt{\frac{U}{\nu x}} = \cancel{0.33206\,\rho\nu U\frac{\sqrt{U}}{\sqrt{\nu x}}}$$

$$= 0.33206\,\rho\nu U\sqrt{\frac{U}{\nu x}} = 0.33206\,\rho U^2\sqrt{\frac{\nu}{Ux}} \quad \rightarrow \frac{1}{\sqrt{Re}}$$

Boundary - Layer characteristics :

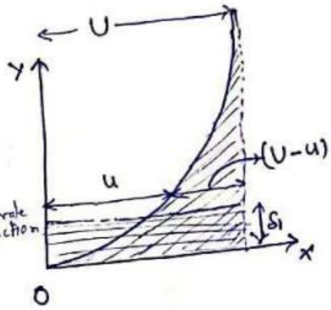
So far we have introduced only one length, δ, which is characteristic of the boundary - layer thickness. This ' total boundary - ~~thic~~ layer thickness' is, however, a nebulous quantity is some respects. For this ~~refon~~ reason and also some other reasons, is it usual to define a small number of lengths, each is a characteristic of the boundary - layer thickness is some way.

(i) Displacement thickness (δ_1) :

Displacement thickness is defined as

$$U \cdot \delta_1 = \int_0^{\infty} (U - u)\, dy \qquad \text{— (1)}$$

decrease in flow rate due to velocity reduction within the B.L.



$$\delta_1 = \int_0^{\infty} \left(1 - \frac{u}{U}\right) dy$$

R.H.S. of (1) is decrease in total flow rate caused by the action of the friction and the ~~left~~ L.H.S represents the flow rate in absence of viscosity, i.e the flow that has been displaced from the wall.

(ii) Momentum thickness (δ_2) : It is defined as

$$\rho U^2 \cdot \delta_2 = \rho \int_0^{\infty} u\,(U - u)\, dy \qquad \text{— (2)}$$

$$\delta_2 = \int_0^{\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

R.H.S of (2) represents the loss of momentum flow ~~as a~~ due to the wall friction in the boundary layer and L.H.S of (2) represents the momentum flow in the absence of the boundary layer.

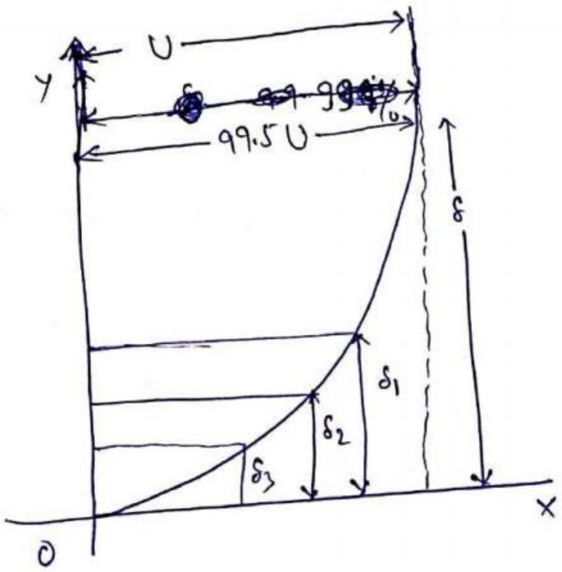
i) Energy thickness (or K.E. thickness) : (δ_3)

It ~~is~~ measures the flux of kinetic Energy loss within the boundary layer as Compared with an inviscid flow.

It is defined as

$$\left(\frac{1}{2}\rho U^3\right)\delta_3 = \frac{1}{2}\rho\int_0^\infty (U^2-u^2).u\,dy$$

$$\delta_3 = \int_0^\infty \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy$$



Comparisom of various thickness of a bounda