$R = \frac{UL}{\nu}$

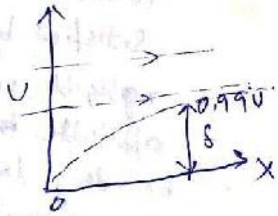
Boundary Layer

So far we have shown a small no. of special cases of flow problems for which exact soln's of N-Stoke's eqn can be found. In those cases the fundamental equations of a viscous fluid happened to be linear because of the simple configurations of the flow patterns and the assumption of incompressible fluid. The solution obtained in the cases mentioned above are valid for all practical values of viscosity. Also when Reylond's no. is very small, the frictional forces are large comparison to the inertial forces and the N-Stoke's eqn's are again linearized, this time by neglecting the convective acceleration. In general, there is no available method yet for finding the soln of the N-Stoke's equations which is valid for all values of viscosity.

In the present topic we consider ~~the~~ flows for very small viscosities and moderate large Reynalds numbers (very small viscocities or ~~very~~ high velocities). ~~will be~~ In this case viscous force are very small compare to inertial forces. Now the question arises: can the soln for incompressible non-viscous flows be regarded as exact solutions of the Navier-Stoke's equations for very large Reynalds no. L. Prandtl ~~in~~ in 1903 first give ~~the~~ answer of the question. He says for high Reynald no. the viscous effects may be neglected everywhere except in a thin ~~layer~~ region in the vicinity of the boundary of a solid. This region is generally referred as Boundary layer into which the effect of viscosity is confined and in which the effect is predominant. The existence of the boundary layer is necessitated by the requirement of the no-slip condition at the walls, a condition which ~~is~~ cannot be satisfied by inviscid flows. Since the viscous forces are not negligible within the B.L. the Euler's eqn of motion is no longer applicable. ~~to the~~ we use N-Stoke's eqn. But the fact that the boundary layers are ~~so~~ very thin permits some simplification of the governing eqn's. The resulting eqn's are called Prandtl boundary layer equations.

Defⁿ: **Boundary Layer Thickness:**

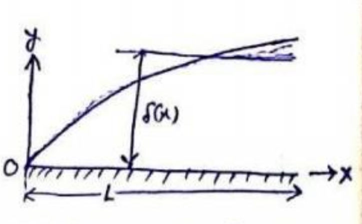
The thickness of the 'velocity boundary layer' is normally defined as the distance from the solid body at which the viscous flow velocity is 99% of the ~~stream~~ free stream velocity.



Prandtl -

~~[illegible]~~ Boundary-layer equations ~~in~~ for 2-Dimensional flow:

Consider a steady 2-D flow of an incompressible fluid / of very small viscosity past a flat plate ~~wedge~~ submerged in the fluid. ~~of very small viscosity~~. At the leading stagnation point O, the thickness of the B.L. is zero and it grows slowly toward the rear of the wedge. The pattern of streamlines and the velocity distribution deviate only slightly from those in potential ~~(frictionless)~~ flow with the exception of the immediate vicinity of the wall. Within a very thin B.L. thickness δ a large velocity gradient exists, i.e the velocity increases from zero at the wall to the value of potential flow at the ~~ed~~ edge of the B.L.



The N-Stoke's eq^n ~~with~~ without body forces for 2-Dim flow are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{—(1)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad \text{— (2)}$$

the continuity eq^n is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{— (3)}$$

Now we will obtain B.L. eq^ns from Eq^n (1), (2) and (3) which are governing eq^n of motion within the B.L. The mathematical procedure for reducing the N-Stoke's eq^n to the B.L. Eq^ns is that of finding the first-order approximation of the former equations when the Reynolds number is very large. In order to compare the order of magnitude of the individual terms in eq^ns (1), (2) & (3) it is more advantageous to put eq^ns in non-dim form by letting

$$x = x^* L, \quad y = y^* \delta, \quad u = u^* U, \quad v = v^* V, \quad t = t^* \frac{L}{U}, \quad p = p^* p_\infty$$

where ~~[illegible]~~ L, U, V and p_∞ are certain characteristic ~~quantities~~ values of quantities x, u, v & p, and δ is B.L. thickness. Note that ~~[illegible]~~ here we are not using L as a ch. length in y direction, because within the B.L. order of lengths along x & y axis are different. So that here [we] have chosen appropriate ch. lengths in both the directions.

the velocity Components in these directions ~~do also~~ have different Characteristic Values. The non-dim. quantities are of order unit, because we have Choosen appropriate ch. values.. If these dimensionless variables are used in Eq^n (3), we get

$$\frac{U}{L}\frac{\partial u^*}{\partial x^*} + \frac{V}{\delta}\frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{\partial u^*}{\partial x^*} + \frac{LV}{\delta U}\frac{\partial v^*}{\partial y^*} = 0$$

~~therefore~~ i.e $\frac{\partial u^*}{\partial x^*} = -\left(\frac{LV}{\delta U}\right)\frac{\partial v^*}{\partial y^*}$

~~i.~~ L.H.S & R.H.S term must be of same order. therefore

$o\left(\frac{LV}{\delta U}\right) = 1$ since ~~o~~ $\frac{\partial u^*}{\partial x^*}$ & $\frac{\partial v^*}{\partial y^*}$ are of order 1.

i.e $o(V) = o\left(U.\frac{\delta}{L}\right)$ —— (4)

Hence $V << U$.

Now we insert non-dimensional Variables in Eq^n (1).

$$\frac{U^2}{L}\frac{\partial u^*}{\partial t^*} + \frac{U^2}{L}u^*\frac{\partial u^*}{\partial x^*} + \frac{VU}{\delta}v^*\frac{\partial u^*}{\partial y^*} = -\frac{p_\infty}{\rho L}\frac{\partial p^*}{\partial x^*} + \frac{\nu U}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{L^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

or
$$\underset{o(1)}{\frac{\partial u^*}{\partial t^*}} + \underset{o(1)}{u^*\frac{\partial u^*}{\partial x^*}} + \underset{o(1)}{\frac{V}{U}\cdot\frac{L}{\delta}v^*\frac{\partial u^*}{\partial y^*}} = -\frac{p_\infty}{\rho U^2}\frac{\partial p^*}{\partial x^*} + \frac{1}{R}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{L^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

~~To~~ Since ~~Re~~ Reynolds no. R is very large, so first viscous term on the R.H.S. of above eq^n is small compared to the order of inertial terms, and hence can be neglected. Since viscous forces must play a role within the B.L., the second viscous term must be retained and thus order of this term must be the order of inertial terms. Thus

$$o\left(\frac{1}{R}\frac{L^2}{\delta^2}\right) = o(1)$$

~~$o(\delta) = o\left(\frac{1}{\sqrt{R}}\right)$~~

i.e $o\left(\frac{\delta}{L}\right) = o\left(\frac{1}{\sqrt{R}}\right)$ —— (5)

This shows that for large R B.L. thickness is very small.
Therefore N-Stoke's Eqn (1) ~~~~ reduces to ~~~~ following eqn within the B.L.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

Again Eqn ② in non-dimensional variables can be written as

$$\underset{o(\delta)}{\frac{V}{U}\frac{\partial v^*}{\partial t^*}} + \underset{o(\delta)}{\frac{V}{U}u^*\frac{\partial v^*}{\partial x^*}} + \underset{o(\delta)}{\frac{V^2}{U^2}\cdot\frac{L}{\delta}\cdot v^*\frac{\partial v^*}{\partial y^*}} = -\frac{p_\infty \cdot L}{\rho U^2 \delta}\frac{\partial p^*}{\partial y^*} + \underset{o(\delta^3)}{\frac{1}{R}\cdot\frac{V}{U}}\left(\underset{o(1)}{\frac{\partial^2 v^*}{\partial x^{*2}}} + \underset{o(\frac{1}{\delta^2})}{\frac{L^2}{\delta^2}\frac{\partial^2 v^*}{\partial y^{*2}}}\right)$$

Clearly order of first term ~~of~~ on the R.H.S of above Eqn is very large compared to the other. Thus we can neglect other terms, the resulting eqn is

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

i.e $\frac{\partial p}{\partial y} = 0$

Hence within the B.L. pressure distribution is a function of x only, i.e their is no pressure variation across the B.L. (valid only in B.L)

Hence B.L. equations ~~for~~ for 2-Dim, flow are,

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \\ &\frac{\partial p}{\partial y} = 0 \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{aligned}\right\}$$

with the B.C.s (i) $y = 0$: $u = v = 0$

(ii) $y = \infty$: $u = U(x,t)$