Summation of Series

To find the sums of type $\sum_{n\in\mathbb{Z}} f(n)$ and $\sum_{n\in\mathbb{Z}} (-1)^n f(n)$ we apply the *Residue Theorem* which is done by means of the following Proposition:

Proposition 1 *Let f be holomorphic on $\mathbb{C}$ except for finitely many points $a_1, a_2, ..., a_k$, none of which is an integer (real). Suppose that there exists $M > 0$ such that $\left|z^2 f(z)\right| \leq M$ for all $|z| > R$ for some R. Let*

$$g(z) = \pi\frac{\cos\pi z}{\sin\pi z} f(z) \quad \textit{and} \quad h(z) = \frac{\pi}{\sin\pi z} f(z) \quad \textit{for all } z.$$

Then

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{j=1}^{k} Res(g, a_j) \quad \textit{and} \quad \sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_{j=1}^{k} Res(h, a_j).$$

Proof. Note that $\sin\pi z = 0$ at each $n \in \mathbb{Z}$ and by hypothesis f is holomorphic at each n. Consider two cases:

Case (i): Let $f(n) \neq 0$ for each n. Then both the functions g and h have simple poles at each $n \in \mathbb{Z}$ and the singularities at $a_1, a_2, ..., a_k$. Let us consider a large rectangle not passing through any integer and containing all singularites $a_1, a_2, ..., a_k$ of f and the integers $-n, ..., -2, -1, 0, 1, 2, ..., n$. Then such rectangle may be the square with vertices $\pm\left(n + \frac{1}{2}\right) \pm i\left(n + \frac{1}{2}\right)$ for large n so that $|a_j| < n$ and that square we denote by S_n. Hence, by Residue Theorem

$$\begin{aligned}\int_{S_n} g(z)\mathrm{d}z &= \sum (\text{Residue of } g \text{ at singularities inside } S_n) \\ &= \sum_{j=1}^{k} \mathrm{Res}(g, a_j) + \sum_{m=-n}^{n} \mathrm{Res}(g, m).\end{aligned}$$

Now, since g has a simple pole at each m,

$$\begin{aligned}\mathrm{Res}(g, m) &= \lim_{z\to m} (z - m)\, g(z) = \lim_{z\to m} \pi\frac{z - m}{\sin\pi z} \cos\pi z\ f(z) \\ &= \pi\left(\lim_{z\to m} \frac{z - m}{\sin\pi z}\right) \cdot \cos\pi m\ f(m) = f(m).\end{aligned}$$

Hence,

$$\int_{S_n} g(z)\mathrm{d}z = \sum_{j=1}^{k} \mathrm{Res}(g, a_j) + \sum_{m=-n}^{n} f(m).$$

Similarly, we have

$$\begin{aligned}\mathrm{Res}(h, m) &= \lim_{z\to m} (z - m)\, h(z) = \lim_{z\to m} \pi\frac{z - m}{\sin\pi z}\ f(z) \\ &= = \pi\left(\lim_{z\to m} \frac{z - m}{\sin\pi z}\right) \cdot\ f(m) = (-1)^m f(m)\end{aligned}$$

as $\cos \pi m = (-1)^m$ and

$$\int_{S_n} h(z)\mathrm{d}z = \sum_{j=1}^{k} \mathrm{Res}(h, a_j) + \sum_{m=-n}^{n} (-1)^m f(m).$$

Case (ii): If $f(m) = 0$ for some m, then g and h have removable singularities at such m and can be taken to be holomorphic there. Hence, in this case also, we have

$$\int_{S_n} g(z)\mathrm{d}z = \sum_{j=1}^{k} \mathrm{Res}(g, a_j) + \sum_{m=-n}^{n} f(m)$$

and

$$\int_{S_n} h(z)\mathrm{d}z = \sum_{j=1}^{k} \mathrm{Res}(h, a_j) + \sum_{m=-n}^{n} (-1)^m f(m).$$

Now, we only need to show that

$$\lim_{n\to\infty} \int_{S_n} g(z)\mathrm{d}z = 0 = \lim_{n\to\infty} \int_{S_n} h(z)\mathrm{d}z.$$

We have

$$\left| \int_{S_n} g(z)\mathrm{d}z \right| \le \pi \int_{S_n} \left| \frac{\cos \pi z}{\sin \pi z} \right| \cdot |f(z)|\, |dz|$$

and

$$\left| \int_{S_n} h(z)\mathrm{d}z \right| \le \pi \int_{S_n} \frac{1}{|\sin \pi z|} \cdot |f(z)|\, |dz|,$$

where by hypothesis, we have for any $z \in S_n$, $|z| > n$ for large n, $|f(z)| \le \frac{M}{|z|^2} < \frac{M}{n^2}$. With the use of the results (which may easily be proved): that for any $z \in S_n$, $|z| > n$,

$$\left| \frac{\cos \pi z}{\sin \pi z} \right| \le A \text{ for some } A > 0$$

and

$$\left| \frac{1}{\sin \pi z} \right| \le B \text{ for some } B > 0,$$

we obtain

$$\left| \int_{S_n} g(z)\mathrm{d}z \right| \le \frac{\pi A M}{n^2} \cdot 4\,(2n+1) \to 0 \text{ as } n \to \infty.$$

Similarly, $\left| \int_{S_n} h(z)\mathrm{d}z \right| \to 0$ as $n \to \infty$. This proves the Proposition.

Remark 1 *If f has singularities at some integer (real) the Proposition above can still be applied. That integer will be excluded from the sum $\sum_{m=-n}^{n}$ and the residue at that integer will be included in the sum $\sum_{j=1}^{k}$.*

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Example 1 *Show that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution 1 *Consider the function* $f(z) = \frac{1}{z^2}$ *which has double pole at* $z = 0$ *and no other singularity (there is no* a_j*). Hence,* $g(z) = \frac{\pi}{z^2} \frac{\cos \pi z}{\sin \pi z} = \frac{\pi \cot \pi z}{z^2}$ *has a pole of order* 3 *at* $z = 0$ *and simple poles at all other integers. The Laurent expansion of* $\cot \pi z$ *at* $z = 0$ *is given by*

$$\cot \pi z = \frac{1}{\pi z} - \frac{\pi z}{3} + ...$$

and hence, the Laurent expansion of $g(z)$ *at* $z = 0$ *is given by*

$$g(z) = \frac{1}{z^3} - \frac{\pi^2}{3} \cdot \frac{1}{z} + ...$$

which shows that $Res(g(z), 0) = -\frac{\pi^2}{3}$*. Thus in view of the above Remark*

$$\sum_{n \neq 0} \frac{1}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} = -Res\,(g(z), 0)$$

which proves that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \left(-Res\,(g(z), 0)\right) = \frac{\pi^2}{6}.$$

Example 2 *Show that, for* $a > 0$ *and not an integer,*

(i)

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a} \coth \pi a.$$

(ii)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 + a^2} = \frac{1}{2a^2} - \frac{\pi}{2a \sinh \pi a}.$$

Solution 2 *Consider the function* $f(z) = \frac{1}{z^2+a^2}$ *which has simple poles at* $z = \pm ai$*. Hence, by Proposition 1,*

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = -Res\,(g, ai) - Res\,(g, -ai)\,,$$

where

$$g(z) = \frac{\pi}{z^2 + a^2} \cot \pi z.$$

Clearly,

$$\begin{aligned} Res\,(g, ai) &= \lim_{z \to ai} (z - ai) \frac{\pi}{(z - ai)(z + ai)} \cot \pi z \\ &= \frac{\pi}{2ai} \cot \pi ai = -\frac{\pi}{2a} \coth \pi a \end{aligned}$$

and

$$\begin{aligned} Res\,(g, -ai) &= \lim_{z \to -ai} (z + ai) \frac{\pi}{(z - ai)(z + ai)} \cot \pi z \\ &= \frac{\pi}{2ai} \cot \pi ai = -\frac{\pi}{2a} \coth \pi a. \end{aligned}$$

Thus

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} + \frac{1}{a^2} = \frac{\pi}{a} \coth \pi a$$

which proves the result (i). To prove the result (ii), we have by Proposition 1,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = -Res\,(h, ai) - Res\,(h, -ai)\,,$$

where

$$h(z) = \frac{\pi}{z^2 + a^2} \frac{1}{\sinh \pi z}$$

and

$$Res\,(h, ai) = -\frac{\pi}{2a} \frac{1}{\sinh \pi a}, \quad Res\,(h, -ai) = -\frac{\pi}{2a} \frac{1}{\sinh \pi a}.$$

hus

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} + \frac{1}{a^2} = \frac{\pi}{a} \frac{1}{\sinh \pi a}$$

which gives

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a} \frac{1}{\sinh \pi a}$$

and this leads the result (ii).

Example 3 *Show that, for $a > 0$ and not an integer,*

(i)

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n + a)^2} = \frac{\pi^2}{\sin^2 \pi a}.$$

(ii)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n + a)^2} = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}.$$

Solution 3 *Consider the function* $f(z) = \frac{1}{(n+a)^2}$ *which has double pole at* $z = -a$*. Hence, by Proposition 1,*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = -Res\,(g,-a)\,,$$

where

$$g(z) = \frac{\pi}{(n+a)^2} \cot \pi z$$

and

$$\begin{aligned} Res\,(g,-a) &= \lim_{z \to -ai} (\pi \cot \pi z)' \\ &= -\frac{\pi^2}{(\sinh \pi a)^2} \end{aligned}$$

hence, the result. Similarly, result (ii) may be obtained.