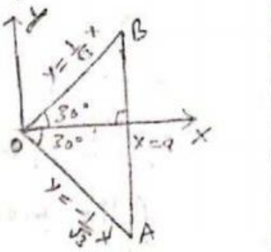
Flow in tube having equilateral triangular Cross-Section :

Let us Consider laminar, steady flow of Incompressible Viscous fluid in a tube of uniform Cross-section. let flow is unaccelerated and there is no body forces.

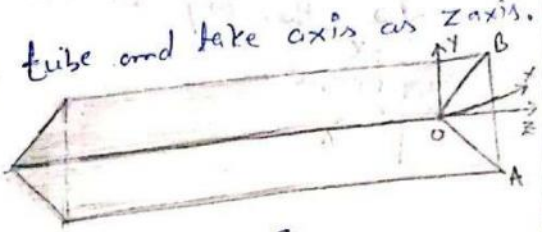


then by Eq^n of Continuity

$$\nabla \cdot \bar{q} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad -(1)$$

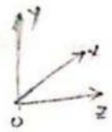
let flow is along the axis of tube and take axis as Z axis.



i.e $\quad \bar{q} = w\hat{k}$

So by (1)

$$\frac{\partial w}{\partial z} = 0 \qquad \text{i.e} \quad w = w(x,y) \quad -(2)$$



Now by N-Stoke's Eq^n of motion for viscous, Incomp. fluid

$$\rho \frac{D\bar{q}}{Dt} = -\nabla p + \mu \nabla^2 \bar{q}$$

Since flow is unaccelerated therefore $\frac{D\bar{q}}{Dt} = 0$ (This is also clear by $u=v=0$ & $\frac{\partial w}{\partial z}=0$)

i.e $\quad \nabla^2 \bar{q} = \frac{1}{\mu} \nabla p$

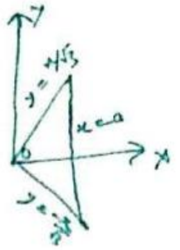
$$\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\hat{k} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right)$$

Comparing coefficients both side we get

$$\frac{\partial p}{\partial x} = 0 \quad ,$$

$$\frac{\partial p}{\partial y} = 0 \quad \text{and}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial z} \quad -(3) \quad (\text{since } w = w(x,y))$$



let $\frac{\partial p}{\partial z} = -P$ (Constant)

So, Eqn (3) Can be Rewritten as

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{-P}{\mu} \quad — (4)$$

Suppose the section of the tube is bounded by the lines

$$x = a, \quad y = \pm \frac{x}{\sqrt{3}}$$

we have to solve Eqn (4) subject to boundary Condition

$w = 0$ on the wall.

we first observe that the function

$$w = k(x-a)\left(y^2 - \frac{x^2}{3}\right) \quad — (5)$$

is appropriately zero on the boundary. To find value k so that w is the required soln of (4) we Substitute w in eqn (4).

$$\frac{\partial w}{\partial x} = k\left[(x-a)\left(-\frac{2x}{3}\right) + \left(y^2 - \frac{x^2}{3}\right)\right]$$

$$\frac{\partial^2 w}{\partial x^2} = k\left[-\frac{2}{3}(2x-a) - \frac{2}{3}x\right] = \frac{2k}{3}(-2x+a-x)$$

$$= \frac{2k}{3}(-3x+a)$$

$$\frac{\partial w}{\partial y} = 2k(x-a)y$$

$$\frac{\partial^2 w}{\partial y^2} = 2k(x-a)$$

Substituiting ω_{xx} & ω_{yy} in (4) we get

$$\frac{2k}{3}(-3x+a) + 2k(x-a) = \frac{-P}{\mu}$$

$$2k\left(-\cancel{x}+\frac{a}{3}+\cancel{x}-a\right) = \frac{-P}{\mu}$$

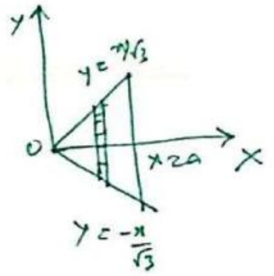
$$2k\left(-\frac{2}{3}a\right) = \frac{-P}{\mu}$$

$$k = \frac{3P}{4a\mu}$$

Thus $\omega = \frac{3P}{4a\mu}(x-a)\left(y^2-\frac{x^2}{3}\right)$

this Sol^n is unique . in virtue of the uniqueness theorem. (Chorlton/334

The volume discharged per unit time is



$$Q = \int_S \omega . dxdy$$

$$= \int_{x=0}^{a}\int_{y=-\frac{x}{\sqrt{3}}}^{\frac{x}{\sqrt{3}}} \frac{3P}{4a\mu}(x-a)\left(y^2-\frac{x^2}{3}\right) dx\,dy$$

$$= \frac{3P}{4a\mu}\int_0^a \left\{2\int_0^{\frac{x}{\sqrt{3}}}(x-a)\left(y^2-\frac{x^2}{3}\right)dy\right\}dx$$

$$= \frac{3P.2}{4a\mu}\int_0^a \left\{(x-a)\left(\frac{y^3}{3}-\frac{x^2y}{3}\right)_0^{x/\sqrt{3}}\right. dx$$

$$= \frac{3P}{2a\mu}\int_0^a \frac{(x-a)}{3}\cdot\left\{\frac{x^3}{3\sqrt{3}}-\frac{x^3}{\sqrt{3}}\right\}dx$$

$$\cancel{= \frac{3P}{2a\mu\sqrt{3}}\int_0^a (x-a)\,x^3\cdot\left(\frac{1}{3\sqrt{3}}-1\right)dx} = \frac{P}{6a\mu}$$

$$Q = \frac{\cancel{3}P}{20\mu} \frac{1}{\cancel{3}\sqrt{3}} \int_0^a -\frac{2}{3}(x-a)\cdot x^3 \, dx$$

$$= \frac{P}{2\cancel{0}\sqrt{3}\,a\mu} \left(-\frac{2}{3}\right) \int_0^a (x^4 - ax^3)\,dx$$

$$= -\frac{P}{3\sqrt{3}a\mu} \left(\frac{x^5}{5} - a\frac{x^4}{4}\right)_0^a$$

$$= \frac{-P}{3\sqrt{3}\,a\mu} \left(\frac{a^5}{5} - \frac{a^5}{4}\right)$$

$$= \frac{-P\,a^5}{3\sqrt{3}\,a.\mu} \left(-\frac{1}{20}\right)$$

$$Q = \frac{P.a^4}{60\sqrt{3}.\mu}$$