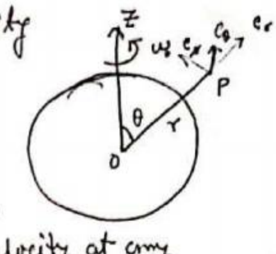
Steady Motion of viscous Fluid Due to a Slowly Rotating sphere

~~For the present Problem, the velocity components are~~



Let sphere is rotating slowly with angular velocity ω_0 about the fixed axis $\theta = 0$ passing through the Centre of sphere. Due to viscosity surrounding fluid will also rotate in the same direction. Thus there will be only motion in the direction of $\hat{e}_\phi$ and $V_r = V_\theta = 0$. Let ω be the angular velocity at any point P. This is to be noted that angular velocity ω is a fn of r only. ~~the components of~~ velocity at P In Cartesian Co-ordinates

~~are~~ $\cancel{u = -\omega y}$ $\cancel{v = \omega x}$ $\cancel{w = 0}$ $\qquad \cancel{\bar{q} = \omega\hat{k} \times (\bar{r})}$

$$\cancel{= \omega\hat{k} \times (x\hat{i} + y\hat{j} + z\hat{k})}$$

$$\bar{V} = \vec{\omega} \times \vec{r}$$

$$= \omega\hat{k} \times (x\hat{i} + y\hat{j} + z\hat{k})$$

$$u\hat{i} + v\hat{j} + w\hat{k} = \omega x\hat{j} - \omega y\hat{i}$$

i.e $\quad u = -\omega y, \quad v = \omega x, \quad w = 0 \qquad \text{— (1)}$

The Navier-stoke's equations in the absence of the body forces is

$$\frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\bar{q} \qquad \text{— (2)}$$

Due to steady motion $\frac{\partial \bar{q}}{\partial t} = 0$. Again, since $\bar{q}$ is very small, So neglecting squares of velocities, we have $(\bar{q}\cdot\nabla)\bar{q} = 0$, Hence (2) reduces to

$$\mu\nabla^2\bar{q} = \nabla p$$

$$\cancel{\mu[\hat{i}\nabla^2 u + \hat{j}\nabla^2 v]}$$

$$\mu[\hat{i}\,\nabla^2 u + \hat{j}\,\nabla^2 v] = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

i.e
$$\left.\begin{aligned} \mu\nabla^2 u &= \frac{\partial p}{\partial x} \\ \mu\nabla^2 v &= \frac{\partial p}{\partial y} \\ 0 &= \frac{\partial p}{\partial z} \end{aligned}\right\} \quad (3)$$

Now we find $\nabla^2 u$ in term of the derivatives w.r.t r. For this we will evaluate

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(-yw) = -y\frac{\partial w}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = -y\frac{\partial^2 w}{\partial x^2}$$

and
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(-yw) = -w - y\frac{\partial w}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial w}{\partial y} - \left(y\frac{\partial^2 w}{\partial y^2} + \frac{\partial w}{\partial y}\right) = -y\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial w}{\partial y}$$

and
$$\frac{\partial u}{\partial z} = -y\frac{\partial w}{\partial z}, \quad \frac{\partial^2 u}{\partial z^2} = -y\frac{\partial^2 w}{\partial z^2}$$

Thus
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$
$$= -y\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - 2\frac{\partial w}{\partial y}$$
$$= -y\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \frac{2}{y}\frac{\partial w}{\partial y}\right] \quad \text{— (4)}$$

~~Now since~~
$$\cancel{\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} = \frac{y}{r}\cdot\frac{\partial w}{\partial r} = \frac{y}{r}\frac{dw}{dr}}$$
$$\cancel{\frac{1}{y}\frac{\partial w}{\partial y} = \frac{1}{r}\frac{dw}{dr}}$$

Now since
$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} = \frac{x}{r}\frac{\partial w}{\partial r} = \frac{x}{r}\frac{dw}{dr}$$
$$\frac{1}{x}\frac{\partial w}{\partial x} = \frac{1}{r}\frac{dw}{dr} \quad \text{— (5)}$$

and
$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\frac{dw}{dr}\right)$$
$$= \frac{1}{r}\frac{dw}{dr} + x\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{dw}{dr}\right)$$
$$= \frac{1}{r}\frac{dw}{dr} + x\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{dw}{dr}\right)\frac{\partial x}{\partial r}$$
$$= \frac{1}{r}\frac{dw}{dr} + \frac{x^2}{r}\cdot\left(\frac{1}{r}\frac{d^2 w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right)$$
$$= \frac{x^2}{r^2}\frac{d^2 w}{dr^2} + \frac{x^2 - r^2}{r^3}\cdot\frac{dw}{dr} \quad \text{— (6)}$$

Similarly

$$\frac{\partial^2 w}{\partial y^2} = \frac{y^2}{r^2}\frac{d^2 w}{dr^2} + \frac{r^2-y^2}{r^3}\frac{dw}{dr} \quad — (7)$$

and $$\frac{\partial^2 w}{\partial z^2} = \frac{z^2}{r^2}\frac{d^2 w}{dr^2} + \frac{r^2-z^2}{r^3}\frac{dw}{dr} \quad — (8)$$

Adding (6), (7) and (8), we have

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{d^2 w}{dr^2} + \frac{3r^2-r^2}{r^3}\frac{dw}{dr}$$

$$= \frac{d^2 w}{dr^2} + \frac{2}{r}\frac{dw}{dr} \quad — (9)$$

~~Using (5) and (9), (4) reduces to~~

Now, $$\nabla^2 u = \nabla^2(-yw) = -\nabla^2(yw)$$

$$= -\left[\frac{\partial^2}{\partial x^2}(yw) + \frac{\partial^2}{\partial y^2}(yw) + \frac{\partial^2}{\partial z^2}(yw)\right]$$

$$= -\left[y\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left(w + y\frac{\partial w}{\partial y}\right) + y\frac{\partial^2 w}{\partial z^2}\right]$$

$$= -\left[y\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial y} + y\frac{\partial^2 w}{\partial y^2} + \frac{\partial w}{\partial y}\right) + y\frac{\partial^2 w}{\partial z^2}\right]$$

By (4)

$$\nabla^2 u = -\left[y\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + 2\frac{\partial w}{\partial y}\right]$$

$$\nabla^2 u = -\left[y\cdot\left(\frac{d^2 w}{dr^2} + \frac{2}{r}\frac{dw}{dr}\right) + 2\frac{y}{r}\frac{dw}{dr}\right]$$

$$\nabla^2 u = -y\left[\frac{d^2 w}{dr^2} + \frac{4}{r}\frac{dw}{dr}\right] \quad — (10)$$

Similarly $$\nabla^2 v = x\left[\frac{d^2 w}{dr^2} + \frac{4}{r}\frac{dw}{dr}\right] \quad — (11)$$

Eqⁿ (3) ~~reduces to~~ bec

with these expressions for $\nabla^2 u$ and $\nabla^2 v$

$$\left.\begin{aligned} -\mu y\left(\frac{d^2 w}{dr^2} + \frac{4}{r}\frac{dw}{dr}\right) &= \frac{\partial p}{\partial x} \\ \text{and} \quad \mu x\left(\frac{d^2 w}{dr^2} + \frac{4}{r}\frac{dw}{dr}\right) &= \frac{\partial p}{\partial y} \\ 0 &= \frac{\partial p}{\partial z} \end{aligned}\right\} \quad (12)$$

All equations in Eq.(12) are satisfied by taking $p=$ constant and

$$\frac{d^2\omega}{dr^2} + \frac{4}{r}\frac{d\omega}{dr} = 0$$

$$\Rightarrow \quad r^4\frac{d^2\omega}{dr^2} + 4r^3\frac{d\omega}{dr} = 0$$

$$\Rightarrow \quad \frac{d}{dr}\left(r^4\frac{d\omega}{dr}\right) = 0 \qquad \text{---(13)}$$

Interating (13),

$$r^4\frac{d\omega}{dr} = c_1$$

$$\frac{d\omega}{dr} = \frac{c_1}{r^4}$$

$$\omega = \frac{-c_1}{3r^3} + c_2$$

$$\omega = \frac{c_3}{r^3} + c_2 \qquad \text{---(14)}$$

If motion is generated by a solid sphere of radius 'a' rotating with angular velocity ω_0. we have B.C.

(i) $\omega = 0$ at $r = \infty$

(ii) $\omega = \omega_0$ at $r = a$.

using these B.C. in (14) we get

$$C_2 = 0$$

$$C_3 = \omega_0 . a^3$$

Thus $\omega = \frac{a^3\omega_0}{r^3}$ ---(15)

* when outer sphere is stationary and is of radius b. then B.C. (i) will be $\omega = 0$ at $r = b$.
we have

$$C_2 = -\frac{\omega_0 a^3}{(b^3-a^3)}$$

$$C_3 = \frac{\omega_0 a^3 b^3}{(b^3-a^3)}.$$

and $\omega = \frac{\omega_0 a^3}{(b^3-a^3)}\left(\frac{b^3}{r^3} - 1\right)$

~~If the motion is due to rotation of two concentric sphere of radius a and b with the angular velocity ω_1~~

~~Here we have.~~

Velocity of fluid Particle in spherical Polar Coordinate is

$$\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,e_\phi$$

here $V_r = V_\theta = 0$ and $V_\phi = \omega r\sin\theta$

The only stress Component is

$$p_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{V_\phi}{r}\right)\right]$$

$$V_r = 0, \quad V_\phi = \omega r\sin\theta$$

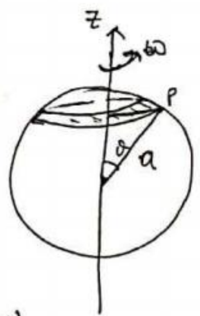
$$p_{r\phi} = \mu r.\frac{\partial}{\partial r}(\omega\sin\theta) = \mu r\sin\theta.\frac{\partial}{\partial r}\left(\frac{a^3}{r^3}\omega_0\right)$$

$$= \mu r.\sin\theta.a^3\omega_0\left(-\frac{3}{r^4}\right)$$

$$p_{r\phi} = -\frac{3\mu a^3\omega_0\sin\theta}{r^3}$$

Stress on the boundary of sphere

$$\left(p_{r\phi}\right)_{r=a} = -3\mu\omega_0\sin\theta.$$



Couple on the sphere

$$= \int_0^{2\pi}(\text{Moment about the z axis}).(2\pi a\sin\theta)(a\,d\theta)$$

$$= \int_0^{2\pi}(p_{r\phi}.a\sin\theta).\,2\pi a^2\sin\theta\,d\theta$$

$$= \int_0^{2\pi} -3\mu\omega_0\sin\theta.\,a\sin\theta\,.\,2\pi a^2\sin\theta\,d\theta$$

$$= -6\mu\pi\omega_0 a^3\int_0^{2\pi}\sin^3\theta\,d\theta = -6\mu\pi\omega_0 a^3.\,2\int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta$$

$$= -12\pi\mu\omega_0 a^3.\,\frac{2}{3} = -8\pi\mu\omega_0 a^3$$