Flow between two ~~Concentric~~ Co-axial Rotating Cylinders:

Consider the ~~#~~ steady flow between two ~~Concentric~~ Co-axial rotating Cylinders with radii r_1 and r_2 respectively. Assume the flow to be peripheral so that we have only tangential Component of velocity V_θ. Let w_1 and w_2 be the steady angular velocities of the inner and outer Cylinders, respectively. also assume that flow is axis symmetric,

By Eqn of Continuity

$$\nabla\cdot\bar{q} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(\cancel{\rho}\, r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\cancel{\rho}\, v_\theta) + \frac{\partial}{\partial z}(\cancel{\rho}\, v_z) = 0$$

Since $V_r = V_z = 0$

So, we have

$$\frac{\partial}{\partial\theta} V_\theta = 0$$

i.e $V_\theta = V_\theta(r)$

The Navier-Stokes Eqn for InComp. with Constant Viscosity fluid and ~~are~~ no body forces are

$$\rho\left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]$$

$$\rho\left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right]$$

$$\rho\frac{Dv_z}{Dt} = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z}$, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$

under our assumptions above Eqns reduces to

$$\rho \frac{V_\theta^2}{r} = \frac{\partial p}{\partial r} \quad \text{— (1)},$$

$$0 = \mu \left(\frac{d^2 V_\theta}{dr^2} + \frac{1}{r}\frac{dV_\theta}{dr} - \frac{V_\theta}{r^2} \right) \text{— (2)},$$

and $$0 = \frac{\partial p}{\partial z} \quad \text{—— (3)}$$

due to axis-symmetry $\frac{\partial p}{\partial \theta} = 0$ & by (3) $\frac{\partial p}{\partial z} = 0$

i.e. $p = p(r)$.

So, Eqn (1) & (2) can be written as

$$\rho \frac{V_\theta^2}{r} = \frac{dp}{dr} \quad \text{— (4)}$$

~~$\frac{d^2 V_\theta}{dr^2} + \frac{1}{r}\frac{dV_\theta}{dr} - \frac{V_\theta}{r^2} = 0$ — (5)~~

and, $$\frac{d^2 V_\theta}{dr^2} + \frac{d}{dr}\left(\frac{V_\theta}{r}\right) = 0 \quad \text{— (5)}$$

Integrating Eqn (5) we get

$$\frac{dV_\theta}{dr} + \frac{V_\theta}{r} = 2C_1$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}(rV_\theta) = 2C_1$$

$$\Rightarrow \frac{d}{dr}(rV_\theta) = 2C_1 \cdot r$$

$$\Rightarrow r \cdot V_\theta = \cancel{2}C_1 \frac{r^2}{\cancel{2}} + C_2$$

$$V_\theta = \frac{C_1}{\cancel{2}} r + \frac{C_2}{r} \quad \text{—— (6)}$$

where c_1 & c_2 ~~are~~ can be Evaluated by using boundary Conditions

(i) $v_\theta = r_1 \omega_1$ at $r = r_1$

(ii) $v_\theta = r_2 \omega_2$ at $r = r_2$

with these B.C. Eq^n (6) gives us

$$r_1 \omega_1 = c_1 r_1 + \frac{c_2}{r_1} \Rightarrow r_1^2 \omega_1 = c_1 r_1^2 + c_2 \quad \text{—— (7)}$$

and $$r_2 \omega_2 = c_1 r_2 + \frac{c_2}{r_2} \Rightarrow r_2^2 \omega_2 = c_1 r_2^2 + c_2 \quad \text{—— (8)}$$

(8) − (7)

$$(r_2^2 \omega_2 - r_1^2 \omega_1) = c_1 (r_2^2 - r_1^2)$$

$$c_1 = \frac{r_2^2 \omega_2 - r_1^2 \omega_1}{(r_2^2 - r_1^2)} = \frac{r_2^2 \omega_2 + (-r_2^2 \omega_1 + r_2^2 \omega_1) - r_1^2 \omega_1}{(r_2^2 - r_1^2)}$$

$$= \frac{(r_2^2 - r_1^2)\omega_1 + r_2^2 \omega_2 - r_2^2 \omega_1}{(r_2^2 - r_1^2)}$$

$$c_1 = \omega_1 + \frac{r_2^2 (\omega_2 - \omega_1)}{(r_2^2 - r_1^2)} \quad \text{—— (9)}$$

also we can ~~find~~ Evaluate

$$c_2 = \frac{-r_1^2 r_2^2 (\omega_2 - \omega_1)}{(r_2^2 - r_1^2)} \quad \text{—— (10)}$$

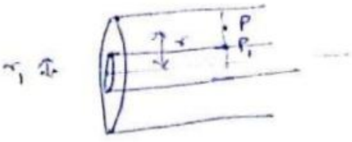
therefore velocity distribution in the pipe is

$$v_\theta = \left\{ \omega_1 + \frac{r_2^2 (\omega_2 - \omega_1)}{(r_2^2 - r_1^2)} \right\} r + - \frac{r_1^2 r_2^2 (\omega_2 - \omega_1)}{(r_2^2 - r_1^2)} \cdot \frac{1}{r} \quad \text{—— (11)}$$

The radial pressure resulting from the peripheral motion ... circular cross section. 1
can be determined from Eqn (4) 4

$$\frac{dp}{dr} = \rho \frac{V_\theta^2}{r}$$

$$\int_{p_1}^{p} dp = \int_{r_1}^{r} \rho \frac{V_\theta^2}{r} dr$$



$$p - p_1 = \rho \int_{r_1}^{r} \left(c_1 r + \frac{c_2}{r}\right)^2 \cdot \frac{1}{r} dr$$

$$p = p_1 + \rho \int_{r_1}^{r} \left(c_1^2 r + \frac{c_2^2}{r^3} + \frac{2c_1c_2}{r}\right) dr$$

$$p = p_1 + \rho \left[\frac{c_1^2}{2}(r^2 - r_1^2) - \frac{c_2^2}{2}\left(\frac{1}{r^2} - \frac{1}{r_1^2}\right) + 2c_1c_2 \log\frac{r}{r_1}\right]$$

where c_1 & c_2 are given by Eqn (9) & Eqn (10).

The shearing stress at any point in this case is

$$\cancel{\tau_{r\theta} = \mu\left[r\frac{d}{dr}\left(\frac{V_\theta}{r}\right)\right]}$$

$$\tau_{r\theta} = \mu\left(\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r} + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right) = \mu\left(\frac{dV_\theta}{dr} - \frac{V_\theta}{r}\right)$$

$$= \mu\, r \cdot \frac{d}{dr}\left(\frac{V_\theta}{r}\right)$$

$$= \mu\, r \cdot \frac{d}{dr}\left(c_1 + \frac{c_2}{r^2}\right)$$

$$= \mu \cdot r\left(\frac{-2c_2}{r^3}\right) = -\frac{2\mu c_2}{r^2}$$

$$= +\frac{2\mu}{r^2} \cdot \frac{r_1^2 \cdot r_2^2 (\omega_2 - \omega_1)}{(r_2^2 - r_1^2)}$$

Shearing stress at the inner and outer walls of the cylinders are

At inner wall

$$(\tau_{r\theta})_{r_1} = \frac{2\mu r_2^2}{(r_2^2-r_1^2)}(\omega_2-\omega_1)$$

stress at the outer wall is

$$= \pm (\tau_{r\theta})_{r_2} = \pm \frac{2\mu r_1^2}{(r_2^2-r_1^2)}(\omega_2-\omega_1)$$